

DOCUMENT RESUME

ED 326 427

SE 051 789

AUTHOR Hope, Jack
TITLE Charting the Course. A Guide for Revising the Mathematics Program in the Province of Saskatchewan. SIDRU Research Report No. 16.
INSTITUTION Regina Univ. (Saskatchewan). Faculty of Education.
SPONS AGENCY Saskatchewan Dept. of Education, Regina.
REPORT NO ISBN-0-7731-0183-7; ISSN-0835-6580
PUB DATE Jul 90
NOTE 326p.
PUB TYPE Information Analyses (070) -- Guides - Non-Classroom Use (055)

EDRS PRICE MF01/PC14 Plus Postage.
DESCRIPTORS Course Content; *Curriculum Development; Curriculum Guides; Curriculum Research; *Elementary School Mathematics; Elementary Secondary Education; Foreign Countries; *Instructional Development; Junior High Schools; *Mathematics Education; *Middle Schools; Science Curriculum; *Secondary School Mathematics; Technology
IDENTIFIERS *Saskatchewan

ABSTRACT

The purpose of this report is to identify changes that need to be made in the Saskatchewan mathematics curriculum and to provide the curriculum writers and advisory committee with sufficient background information to develop a revised program and to evaluate teaching materials currently available. The purpose of Part 1 of the guide is to examine the goals of mathematics education in more detail. Chapters 1 and 2 discuss how the study of mathematics can prepare students to function in society as consumers and employees, and as informed and responsible citizens. Chapter 3 examines the argument that mathematics plays a significant role in developing a liberally educated people: those who can think logically, possess good working habits, and appreciate the cultural and aesthetic merits of mathematics. Chapter 4 concludes Part 1 by examining the effect of technology on the need to include traditional algorithmic knowledge in the mathematics curriculum. The intent of Part 2 is to provide the curriculum writers with a good background in the issues surrounding curriculum development in mathematics and in research related to student understanding of mathematics. Chapter 5 provides an overview and critique of the current elementary and middle years mathematics programs. Chapters 6 through 14 discuss the following mathematical areas as they apply to a revised elementary and middle years program: "Problem Solving"; "Foundations of Number"; "Basic Number Combinations"; "Computational Knowledge"; "Estimation"; "Measurement"; "Geometry and Spatial Sense"; "Data Organization and Analysis"; and "Algebra." A discussion of the current secondary mathematics program is presented in chapter 15 and recommendations for revising the secondary mathematics program are proposed in chapter 16. Included are over 140 references. (KR)



SASKATCHEWAN INSTRUCTIONAL DEVELOPMENT & RESEARCH UNIT

ED326427

CHARTING THE COURSE

A Guide for Revising the Mathematics Program
in the Province of Saskatchewan

Researcher/Writer:

Jack Hope, B.Sc., M.A., Ed.D.

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

☒ This document has been reproduced as
received from the person or organization
originating it.

☐ Minor changes have been made to improve
reproduction quality.

• Points of view or opinions stated in this docu-
ment do not necessarily represent official
OERI position or policy.

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY

Paul Hart

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

SIDRU Research Report No. 16
July 1990

CHARTING THE COURSE

A Guide for Revising the Mathematics Program in the Province of Saskatchewan

**Prepared for the
Saskatchewan Instructional Development
and Research Unit**

**With Funding from
Saskatchewan Department of Education**

**by
Jack Hope, B.Sc., M.A., Ed.D.
July 1990**

Published 1990 by the Saskatchewan Instructional Development and Research Unit with funding assistance from Saskatchewan Department of Education. Publication does not constitute endorsement of the findings or opinions by either of these agencies.

Permission to reproduce or transmit any part of this report must be obtained in advance in writing from the Saskatchewan Instructional Development and Research Unit, University of Regina, Regina, Saskatchewan, Canada S4S 0A2.

ISBN 0-7731-0183-7

ISSN 0835-6580

SIDRU Research Report No. 16

K-12 MATHEMATICS ADVISORY COMMITTEE

Mr. Mike Fulton, Chair	League of Educational Administrators
Dr. Dennis Hanson	Department of Mathematics and Statistics University of Regina
Dr. Edward Klopouschak	Faculty of Education University of Regina
Dr. Garth Thomas	Department of Mathematics University of Saskatchewan
Dr. James Beamer	College of Education University of Saskatchewan
Ms. Sharon Kvinlaug	Saskatchewan Institute of Applied Science and Technology
Mr. Harold Flett	Indian and Metis Educational Advisory Committee
Ms. Lillian Gauthier	Official Minority Language Office
Mr. Bob Kallio	Saskatchewan School Trustees Association
Ms. Ruth Taylor	Saskatchewan School Trustees Association
Mr. Laurence Owen	Saskatchewan Teachers Federation
Ms. Verda Petry	Saskatchewan Teachers Federation
Ms. Heather Wright	Saskatchewan Teachers Federation
Mr. Thom Koroluk	Saskatchewan Teachers Federation
Ms. Mary Reeves	Saskatchewan Teachers Federation
Mr. Derek Punshon	Saskatchewan Teachers Federation
Ms. Judy Wall	Special Education
Mr. Barry Mitschke	Saskatchewan Education

TABLE OF CONTENTS

PREFACE	i
PART 1: THE GOALS OF MATHEMATICS EDUCATION	1
CHAPTER 1: MATHEMATICAL PREPARATION OF CONSUMERS AND WORKERS	5
The Mathematics of Daily Living	6
Research into Off-the-job Mathematics	6
Summary of Off-the-job Mathematics	10
The Mathematics of Employment	11
The Mathematics of Non-technical Careers	11
The Mathematics of Technical Careers	14
Summary of Employment Mathematics	16
CHAPTER 2: MATHEMATICAL PREPARATION OF INFORMED & RESPONSIBLE CITIZENS	21
Interpreting Statistical Information	21
Interpreting Chances and Risks	28
CHAPTER 3: MATHEMATICS AND LIBERAL EDUCATION	33
Logical Thinking	33
Culture and Aesthetics	36
CHAPTER 4: TECHNOLOGY AND MATHEMATICS	39
Calculators and the Curriculum	39
Computers and the Curriculum	42
PART 2: REVISING THE SASKATCHEWAN MATHEMATICS PROGRAMS	47
CHAPTER 5: REVISING THE ELEMENTARY AND MIDDLE YEARS PROGRAMS	49
An Overview of the Current Elementary Program	49
An Overview of the Current Middle Years Program	54

Determining New Program Goals	57
Recommendation 1	58
Recommendation 2	59
Recommendation 3	61
Recommendation 4	62
Recommendation 5	65
Recommendation 6	66
Recommendation 7	67
Recommendation 8	68
Recommendation 9	68
 CHAPTER 6: PROBLEM SOLVING	 71
Translation Problems	72
Concept Development	78
Student-Created Problems	80
Reading of Problems	80
Semantic Structure	83
Change Problems	84
Combine Problems	84
Compare Problems	84
Process Problems	96
Steps in Solving Problems	97
Stage 1. Understanding	98
Stage 2. Planning	99
Stage 3. Reflecting	103
Presenting a Mixture of Process Problems	104
Realistic Problems	106
 CHAPTER 7: FOUNDATIONS OF NUMBER	 115
Part-Whole Relationships	115
Counting	116
Place Value	119
Fractions	123
Ratio and Proportion	126
The Unit Method	129
The Common Multiple Method	130
Percentage	131

CHAPTER 8: BASIC NUMBER COMBINATIONS	.137
Shifting Expectations of Memorization	.138
Reconstruction Strategies	.143
Addition Strategies	.143
Subtraction Strategies	.145
Multiplication Strategies	.147
Division Strategies	.149
Promoting and Maintaining Recall	.150
CHAPTER 9: COMPUTATIONAL KNOWLEDGE	.155
Pencil-and-Paper Calculation	.156
Standards of Performance	.156
The Development of Understanding	.157
Electronic Calculators	.159
Mental Calculation	.161
Guidelines for Instruction and Program Development	.165
Conceptualizing	.165
Formalizing	.165
Practising	.165
Extending	.166
Additional Considerations	.166
A List of Mental Calculation Strategies	.168
CHAPTER 10: ESTIMATION	.175
Conceptual Components	.175
Role of Approximation	.175
Multiple Process and Multiple Outcomes	.178
Appropriateness of an Estimate	.179
Skill Components	.180
Reformulation	.180
Compensation	.181
Translation	.181
Related Concepts and Skills	.182
Powers of 10	.182
Comparison	.182
Mental Computation and Recall of Basic Facts	.182
Number Principles	.183
Number Modification	.183
Unit comparison	.183

Basic Number Equivalents	184
Benchmark Equivalents	184
 CHAPTER 11: MEASUREMENT	187
The Measurement Process	187
Measurement as Perception	188
Measurement as Comparison	188
Measurement as Quantification	188
The Understanding of Length	191
The Understanding of Area	194
The Understanding of Volume	195
The Understanding of Mass	197
The Understanding of Time	198
The Understanding of Temperature	200
The Understanding of Angles	201
 CHAPTER 12: GEOMETRY AND SPATIAL SENSE	205
The Goals of Geometry Instruction	206
Everyday Needs	206
Developmental Needs	206
Subject-matter Goals	208
Meeting Process Goals	208
Developing a Framework for Instruction in Geometry	212
Planning a Geometry Curriculum	216
Applying van Hiele Based Research	216
Applying Spatial Perception Research	218
 CHAPTER 13: DATA ORGANIZATION AND ANALYSIS	225
Collecting Data	226
Simple Experimentation	226
Survey Research	227
Published Compiled Data	228
Organizing and Displaying Data	228
Summarizing Data	234
Interpreting Data	235

CHAPTER 14: ALGEBRA	239
The Goals of Algebra Instruction	239
The Understanding of Algebra	241
Variables	241
Word Problems	245
Equations and Formulas	246
Algebraic Expressions	251
Functions	252
CHAPTER 15: THE CURRENT SECONDARY MATHEMATICS PROGRAM	259
The Academic Mathematics Courses	260
General Aim	260
The Components and the Content	261
The Alternative Mathematics Courses	264
General Aim	264
The Components and the Content	265
Appraising the Secondary Mathematics Program	266
CHAPTER 16: REVISING THE SECONDARY MATHEMATICS PROGRAM	279
Establishing Realistic Goals	280
Preparing Students for a Technological Age	280
Preparing Numerate Students	281
Preparing Liberally Educated Students	281
Preparing Students for Careers	282
Developing a Program Structure	283
Structure 1: A Modification of the Current Structure	283
Designing Program A	287
Designing Program B	287
Barriers to the development of a dual program	290
Structure 2: The Structures of B.C. and Alberta	297
The British Columbia program	297
The Alberta program	301
Structure 3: A Common Multiple Exit Program	304
REFERENCES	311

PREFACE

Based on the results of a review of education in Saskatchewan initiated by Saskatchewan Education in 1981, the Core Curriculum Advisory Committee (1986) proposed a policy framework for Saskatchewan schools. In addition to a set of six Common Essential Learnings (CELs), whose purpose is to provide "generic skills, processes, values, and cognitive abilities which can be applied in a wide range of settings and situations," several Required Areas of Study (RASs), including mathematics, were considered by the committee to comprise the "core curriculum."

In 1987, the report Core Curriculum Plans for Implementation stated that writers working with curriculum advisory committees would develop a policy report for each required area before serious program revision would begin. This report is intended to set the stage for changing the Saskatchewan mathematics program in the elementary, middle, and secondary grades. Although the report does not provide a detailed list of learning outcomes for each grade, the key elements to be considered in revising the entire mathematics curriculum are specified. There is sufficient detail in this report to guide the curriculum writers responsible for developing, implementing, and evaluating the mathematics program.

Curriculum development in mathematics education involves much more than selecting topics to be studied, deciding on their order of presentation, and then identifying the currently most appropriate textbook series. Curriculum development must detail not only what mathematics students need to know but also the reasons for acquiring this knowledge. Given the explosion of knowledge in mathematics - claims have been made that over half of all mathematics knowledge has been developed since the second World War - it is clearly impossible to expect anyone to learn everything about the field. This abundance of knowledge means that content must be selected judiciously, especially at the secondary school level, where consensus about the goals of mathematics education and the mathematical content to be studied is difficult to reach.

Curriculum development must also take into account the nature of the learner, the nature of the teacher, and the nature of mathematical knowledge itself. Mathematical content should not be separated from the way it is to be taught and the way children learn.

It became quite clear as I wrote this report that for reasons of time rather than of disinterest I would have to ignore some general but important issues related to the development of a new mathematics curricula: gender and mathematics, French Immersion, mainstreaming, cross-cultural education, teacher education, and so forth. Unquestionably these issues will have to be considered by Saskatchewan Education

as new programs are being developed. I have not written a separate chapter on the Common Essential Learnings. Instead I have attempted to make an occasional reference throughout this report to the CELs, especially Critical and Creative Thinking, Communication, and, of course Numeracy.

This report recognizes that various economic, social and political realities will have to be faced by teachers as they implement the changes proposed in this report. Contrary to the popular view that extensive changes have been made to the mathematics curriculum, relatively few have been made over the last 50 years. The mathematical content studied in schools and the instructional practices of mathematics teachers have, for the most part, remained the same. Based on his analysis of several large scale surveys and detailed case studies, Fey (1981) suggests one reason for the lack of change in American mathematics education, as well as the pessimism of educators regarding major change in the near future:

Despite highly publicized criticism of school mathematics, a large majority of people believe current high school programs are satisfactory. The survey follow-up to the case studies found that over 90 percent of all administrators, supervisors, teachers, students, and parents judged the overall quality of the high school mathematics program as either satisfactory, very good, or excellent. We found this response to be among the most paradoxical of all survey findings. It appears that even though teachers, students, teacher educators, and curriculum developers see serious problems in mathematics education, they are satisfied with their current efforts - perhaps reasoning that immovable boundary conditions limit their potential effectiveness. Whereas there are undoubtedly many mathematics teachers who do not share the enthusiasm for current programs, they face a tremendous challenge in stimulating and carrying out any major change. (p. 29)

Regardless of the obstacles confronting curriculum writers and teachers, the Saskatchewan mathematics program needs to be changed. The purpose of this report is twofold. First, it will identify the changes that need to be made. Second, it will provide the curriculum writers and advisory committee with sufficient background information to develop a revised program and to evaluate teaching materials currently available.

I personally want to express my appreciation to the members of the Mathematics Advisory Committee for their assistance over the past two years. I was greatly aided by their willingness to read the many drafts of the report, to express their views candidly yet forcefully in our many meetings, and to give me the freedom to develop the report as I desired. I am also indebted to Helen Horsman for her willingness to plow through the final draft of the report and to my excellent staff, Beatrice Leer and

Theresa Braun, who uncomplainingly corrected the text when it was transformed from my word processor to the office word processor. I also thank Saskatchewan Education for the opportunity to work on this project and am indebted to Barry Mitschke for his patience and support. I would like to thank my family, Charlene, Shaun and Kim, for their support and encouragement over the past two years as I wrote this report. Finally, I would like to dedicate this report to the memory of my recently deceased father whose sacrifices and support for me will never be forgotten.

Jack Hope
University of Saskatchewan

PART 1

THE GOALS OF MATHEMATICS EDUCATION

Mathematics contains much that will neither hurt one
if one does not know it nor help one if one does know it.

Anonymous

There is general agreement that every child should study mathematics at school. The study of mathematics - together with the study of English - is regarded by most people as being essential. The value of mathematics instruction was clearly recognized in a United States Gallup survey conducted a few years ago. For the first time in its history of polling public opinions, Gallup found that the average person considers mathematics, rather than English, to be the most important school subject. The public seems to believe that the continual well-being of our modern society depends largely on producing school graduates who have studied a great deal of mathematics, at least as much as students in other industrialized countries.¹

Because of the close association between mathematics and the computing sciences, the study of mathematics is viewed by many as a stepping stone to a rewarding and prosperous career. It is also believed that those who do not study a great deal of mathematics will be unable to compete or function in a computerized, technological society. Are these public perceptions about the importance of the study of mathematics realistic? If mathematics must be studied in school, what should students be expected to learn? Part 1 of this paper addresses these important questions in a discussion of the goals of school mathematics.

Three reasons are commonly presented to the public to gather support for the required study of mathematics in school. First and foremost, mathematics is considered to be useful; consequently, its study fills an important social need. Most people tend to see mathematics in terms of the arithmetical knowledge needed to function in the everyday world as consumers and homemakers. Other people, particularly employers who hire school graduates, see mathematics in terms of those skills and abilities needed to function in a work setting. They expect schools to produce knowledgeable and industrious workers who need little additional training. Persons who believe that education should develop an informed and responsible citizenry see mathematics as a means of understanding and recognizing the assumptions underlying arguments and interpretations based on quantitative information. Other people, particularly those teaching in universities and technical colleges, see mathematics as

a tool for understanding the advanced techniques associated with the physical sciences, engineering, computing sciences and most other technical fields.

Although few people today would dispute the usefulness of mathematics, there is little agreement on how much and what type of mathematics will meet the needs of all students. As computers handle many more of the routine mathematical tasks presently taught in school, even less agreement can be expected in the future. There seems to be consensus, however, that the primary goal of mathematics instruction is to prepare students to carry out practical tasks and solve practical problems.

The second justification for the study of mathematics involves the abilities and habits it supposedly fosters: the ability to work with abstractions, carry out step-by-step symbolic manipulations, recognize logical relationships and patterns, discover and formulate generalizations, and reduce complex situations into simpler parts. The study of mathematics, according to this view, helps discipline the mind, promotes perseverance and contributes to the development of higher-order reasoning skills, including critical thinking.

The third reason used to justify the study of mathematics is its cultural and aesthetic merits. The study of mathematics is seen as fostering an appreciation of the beauty and elegance of mathematics, as seen in the geometrical forms prevalent in nature and in art and the logical forms of mathematical proofs. It is also thought that students should learn to appreciate the power of mathematics and the role that mathematics and abstract thinking have played in the development of civilization. Knowledge about the structure of the mathematics discipline, specifically its epistemology, is considered by many to be an important goal of mathematics education.

The general goals and views about the purpose of mathematical studies influence many aspects of the school mathematics curriculum. These include the content studied, the instructional materials used, the instructional approaches taken by teachers, and the assessment of student achievement. The debate that often surrounds mathematics education can be largely attributed to differing views about the purposes of studying mathematics in school. A parent who believes that the purpose of school mathematics is to prepare students for entering the workforce will question the appropriateness of any topic seen as unneeded in coping with quantitative situations arising in everyday life. The suitability of topics such as trigonometry, algebra, and Euclidean geometry, which are the very essence of the traditional secondary school mathematics program, are often questioned. A university science, mathematics or engineering professor who believes that the major purpose of school mathematics, especially as studied in secondary schools, is to prepare students to study mathematics at an advanced post-secondary level will feel compelled to resist attempts to make the content more practical for the average student. University faculty usually view changes to make a program more practical for the majority of students not

intending to study mathematics after graduating from high school as a "lowering of standards." Traditionalists will resist the excision of content from the traditional curriculum, even though it is outdated and rarely used outside of school. They believe a part of our cultural heritage will be lost, leading to a further dilution in the quality of education.

Educational change is a complex enterprise and advocates of change frequently find themselves envisioning a curriculum that is more aspiration than reality. Before a serious revision of the mathematics program can be undertaken, the curriculum developers will have to consider carefully the goals of mathematics education.

The purpose of Part 1 is to examine the goals of mathematics education in more detail. Chapters 1 and 2 discuss how the study of mathematics can prepare students to function in society as consumers and employees, as informed and responsible citizens, and, for a small minority, as scientists and technicians. Chapter 3 examines the argument that mathematics plays a significant role in developing liberally educated people: those who can think logically, possess good working habits, and appreciate the cultural and aesthetic merits of mathematics. Chapter 4 concludes Part 1 by examining the effect of technology on the need to include traditional algorithmic knowledge in the mathematics curriculum.

Footnotes

- ¹ Even adults who presently use little mathematics in their daily lives and who disliked mathematics when they studied it in school seem to share this belief.

CHAPTER 1

MATHEMATICAL PREPARATION OF CONSUMERS AND WORKERS

Since the creation of formal schooling, mathematics has been considered a key subject. The four liberal arts - the quadrivium, consisting of arithmetic, geometry, astronomy, and music - are essentially mathematical in nature. A classical education was not intended initially for the masses of workers, but for only the elite few who were destined to govern. Consequently, mathematical study was not originally intended to be a practical subject designed to prepare one to earn a living.

When compulsory schooling was introduced, the utility of mathematical study became a considerably more important issue than it once had been. Instrumentality has been and will continue to be the major reason for including mathematics as a required school subject and for its relatively high status among other school subjects.

If school mathematics programs are to prepare students to meet the practical needs of society, there must be a clear conception of what these needs currently are and what they are likely to be in the future. Curriculum developers find it difficult to identify these needs because what is useful today may not necessarily be useful tomorrow. For example, logarithms were once essential for performing complex calculations and much study in secondary mathematics programs was devoted to learning to calculate with them. Since the introduction of the inexpensive scientific calculator, learning to calculate with logarithms is not a reasonable goal of instruction, although logarithms continue to be studied as an important type of mathematical function.

In The Saber Tooth Curriculum, a parable of a primitive society, Benjamin (1939) identifies the problems that occur when times change more quickly than curricula. Initially, only the truly useful and essential subjects are taught in the prehistoric, saber-tooth curriculum. For example, the prehistoric students are taught "woolly-horse-clubbing" so they can provide the hides needed for clothing. Although the prehistoric world changes and woolly horses eventually become extinct, the schools do not readily adapt. They continue to teach the now out-dated and irrelevant subjects of woolly-horse-clubbing, fish-grabbing-by-the-bare-hands, and so forth. To ward off critics with their proposals to reform the curriculum, the educators argue that the social utility of these school subjects is not an important consideration because "You must know that there are some eternal verities, and the saber-tooth curriculum is one of them" (p. 44). In this chapter, I will argue that much of the mathematical content

traditionally studied in school has gradually lost its usefulness, and like woolly-horse-clubbing, continues to be studied for reasons other than its utility.

The Mathematics of Daily Living

Hardly a day passes by in most of our lives when some form of mathematical idea is not used routinely. We regularly carry out many tasks on and off the job effortlessly and with little conscious thought when we count objects, read clockfaces, count money, read prices, make a measurement, calculate a simple sum, locate a page in a book or magazine, write a cheque, consult a calendar, interact with an automatic bank teller, and so forth.

Most commonly seen mathematical tasks require no more knowledge than what has been traditionally taught in the elementary school. But many people would argue that we all need to learn more mathematics than these rudimentary understandings to function in an increasingly complex and changing technological age. Contrary to what many people believe about the importance of mathematics, there is compelling evidence to indicate that few people use the mathematical procedures, routines, and exercises taught in school. Moreover, when mathematics is used, the approaches often do not resemble those taught in school.

Research into Off-the-job Mathematics.

A British study sponsored by the Advisory Council for Adult and Continuing Education (1982) - the ACACE study - investigated the use of mathematics in daily life by interviewing adults of both sexes, and of varying ages, educational backgrounds and occupations. Each participant's proficiency in basic mathematics - mainly notions of arithmetic - was also assessed informally.

To tease out some of the number knowledge and skills that each person used regularly, the ACACE study presented participants with several quantitative situations likely to be familiar and asked them to discuss their typical behaviour in these practical situations. For example, people were asked to explain how they would know if they had enough money to pay for groceries in a shopping cart or how they would calculate a price markdown of 10%.

Those interviewed varied considerably in their willingness and ability to handle the common problems used in the ACACE study. For example, about one third of the sample thought that all percentages were meaningless, including straightforward ones like 10%. The following replies typified the adult's views about percentages: "They're beyond me"; "You've got me there"; and "I'm completely lost with percentages."

There appeared to be no relationship between the extent to which adults used mathematics and the extent of their formal education. In the ACACE study, there were people with science degrees who claimed to use no arithmetic. In contrast, there were those with no qualifications who displayed a high level of mathematical competence.

In interviews with a subsequently smaller sample, the adults were questioned more intensively on their handling of a wide range of everyday situations requiring seemingly straightforward mathematics. Original documents such as bills, wage slips, and timetables were often used in these simulations of realistic tasks. For example, the participants were shown an actual printed wage slip for a person named Jennifer. They were asked to use the information in the 25 labelled boxes to determine the total deductions from her pay. All participants were free to use any calculation method or device, including mental methods, pencil-and-paper methods, and calculators.

Although the adults seemed to have little difficulty in producing reasonable answers to most of the questions, many had difficulty with problems involving percentages, ratios, graphs and charts, timetables, metric units, and metric/Imperial conversions. The majority of the sample tested could not handle the following tasks: (1) use a timetable to determine the arrival time in a station of a train departing from another station; (2) compare prices of gasoline selling in litres or gallons; (3) provide a reasonable estimate for the amount of sugar needed to make jam using 3 lbs. of strawberries if the recipe called for $3\frac{1}{2}$ lbs. sugar for every 4 lbs. of strawberries; (4) calculate fuel consumption given the initial odometer reading and fuel gauge readings, two "fill-up" quantities, and the final odometer and gas gauge readings.¹

The most difficult problem involved the notion of inflation. Less than one third gave a reasonable answer when asked to predict the direction of prices if the inflation rate fell from 17.4%, to 17.2%, and to finally 12%. Most adults concluded erroneously that a fall in the rate of inflation would result in the lowering of prices. The ACACE enquiry officer responsible for this study was especially distressed by this "alarmingly widespread (and confident) misunderstanding of inflation revealing a lack of comprehension of the most commonly quoted feature of the national economy. The mathematical aspect of the question on inflation - referring to a change of rate, like easing off on the accelerator - scarcely entered into the replies" (p. 56).

To determine if the difficulties encountered by the ACACE sample were representative of the British population, ten questions similar to those used in the interviews were included in a Gallup survey of 3000 adults. The results of the larger survey indicated that the findings of the smaller ACACE study were typical. For example, only 40% of the Gallup sample knew prices would still rise if inflation dropped from 20% to 15%.

The ACACE study also revealed that much of the computation laboriously practised in school is either forgotten or else its relevance to everyday problems is never perceived. To compensate for the absence of a school-like procedure, many adults develop other computational alternatives such as computing differences by adding on instead of subtracting. The methods relating to mental arithmetic are clearly useful and important to most adults, although rarely developed in school.

In contrast to the traditional notion fostered in schools of a single best or mathematically superior method for solving problems, there is considerable variation in the methods adults use to solve problems. There is also a surprising reluctance, especially by women, to estimate. Most adults seem to assume that only exact answers are acceptable, even when tasks call for approximate answers. Perhaps many adults view estimates as "second-rate" answers or confuse them with haphazard guessing.

The ACACE study also found that many adults are either unable or unwilling to use a calculator, even though calculators are in almost every home. Often people complain that they are confused by and find it difficult to interpret the large number of extraneous digits presented in the display. Some people refuse to use a calculator because of a fear of becoming dependent upon a machine. There are many adults, however, whose computational skills are weak and the calculator enables them to function confidently and competently.

Perhaps the most striking feature of the ACACE study is the feelings expressed by adults when they were asked to perform some seemingly straightforward mathematics. Feelings of anxiety, helplessness, fear and, in some cases, guilt were reported to the ACACE interviewers. The large number of adults who were originally asked to participate in the study but refused to do so may have reflected apprehension about having their mathematical prowess evaluated. The enquiry officer reported that "both direct and indirect approaches were tried, the word 'mathematics' was replaced by 'arithmetic' or 'everyday use of numbers', but it was clear in many cases that the reason for people's refusal to be interviewed was simply that the subject was mathematics" (p. 32).

The ACACE study as well as other studies (Capon & Kuhn, 1979) suggest that many adults either avoid or cannot handle some types of everyday problem solving. These studies used simulations of everyday practice conducted in environments controlled by a researcher. Therefore, they may not provide the same type of information about the problem-solving practices of people as they are engaged in actual day-to-day quantitative situations.

Recently, a number of studies have investigated the arithmetic practices of people as they function in naturally occurring quantitative situations. Murtaugh (1985) studied grocery shoppers in a supermarket; De la Rocha (1985) studied dieters preparing

meals in their kitchens; Scribner (1985) studied commercial dairy workers as they went about their work; Carraher, Carraher & Schliemann (1985a & 1985b), went as customers to Brazilian open air markets to observe young children handling the arithmetic involved in the selling of unusual quantities of fruit. All participants in these studies were later tested on arithmetic problems and procedures supposedly needed to carry out the routine tasks encountered in the out-of-school world. While only Murtaugh's study will be reviewed in some detail here, little generalization will be lost because this study's methodology and findings are similar to the other cited studies, despite the great variation in the settings and sample characteristics.

Murtaugh found that all shoppers, with the exception of one person, made some use of arithmetic while grocery shopping. As might be expected, most of the supermarket arithmetic involved price comparisons, usually straightforward ones such as comparing the prices of two products of equal or approximately equal sizes of quantities.

The most complex supermarket arithmetic involved best-buy comparisons where items were packaged in different amounts or sizes. About 30% of all supermarket calculations involved these type of comparisons. Interestingly, although unit prices were visibly evident in the supermarkets, they were generally ignored by the shoppers in the study. Shoppers rarely calculated unit prices to make comparisons and, when price comparisons were made, no one set up a series of equivalent proportions as is often the practice emphasised in schools. Instead, two procedures were used frequently by the shoppers. When the ratios of the compared quantities were 2:1 or 3:1, prices were compared by doubling or tripling the price of the smaller size. This method is not only easier to do mentally than unit price calculations, but it also provides a shopper with information about the amount saved. For example, one shopper compared two boxes of sugar, one priced at \$2.16 for 5 pounds, the other at \$4.30 for 10 pounds. The shopper explained: "The five pounds would be four dollars and 32 cents, versus four dollars and 30 cents. I guess I'm going to have to buy the ten-pound bag just to save a few pennies" (p. 190). In this case, a unit price calculation would reveal that the larger size costs 0.2 cents less per pound than the smaller size, meaningless information for most shoppers.

When the size comparison ratio was not exactly 2:1, the shoppers still used the doubling procedure effectively. For example, one shopper compared two cans of tomato sauce containing 15 ounces for 39 cents and 8 ounces for 18 cents. The shopper decided to buy two cans of the smaller size by reasoning, "That's 36 cents for 16 ounces, and the bigger is 39 for 15" (p. 190). To this shopper, it was more relevant and convenient to consider relative price differences than to compute and compare two unit prices: in this case, comparing $39/15 = 2.6$ cents and $36/16 = 18/8 = 2.25$ cents.

Summary of Off-the-job Mathematics.

Based on the findings of these studies of everyday off-the-job mathematics, several conclusions can be made. The most obvious conclusion is that most people neither use nor need much mathematics to function properly in ordinary life. As technology continues to degrade previously valued knowledge and skills, the mathematical requirements of future life might be as minimal as suggested by de Bono (1979): simple addition and subtraction; the use of decimal notation; multiplication and simple division; the use of a pocket calculator; the handling of elementary percentages; the mathematics of measurement at a basic level; and the use of computers (p. 213).

Certainly, students need to study mathematics beyond that used in everyday life. Usefulness is not the sole reason for selecting content for study in school. But mathematics educators must appreciate that large numbers of teachers, administrators, parents and even many students do not understand why mathematics should be studied when much of the content is never seen and used outside of school.

The trend of technology to degrade traditional algorithmic knowledge, such as computation and algebraic manipulation, will continue unabated. Consequently, unless there are sound reasons for doing so, requiring children to master arithmetic procedures that have become, or will soon become, automated is wasteful, futile, and possibly hypocritical. To argue that the average person needs to learn a great deal of modern mathematics to cope with the mathematical demands of everyday life in the so-called information age seems unrealistic and, perhaps, even self-defeating.

Problems and tasks encountered outside of school seem to differ considerably from those seen in school. In the world of the adult, people tend to select mathematical techniques according to a number of criteria: the purpose for pursuing a quantitative approach; the size and type of numbers involved; the degree of precision and accuracy required; and, most importantly, helpful contextual cues in the environment. For example, shoppers use a variety of ways to compare prices rather than applying a single, school-learned procedure. Dieters use handy tools like measuring cups to calculate fractional amounts instead of resorting to pencil-and-paper arithmetic. For example, in halving a recipe calling for $\frac{3}{4}$ cup of milk, kitchen math usually involves pouring milk halfway to the $\frac{3}{4}$ mark on a measuring cup rather than multiplying $\frac{1}{2}$ by $\frac{3}{4}$ or dividing $\frac{3}{4}$ by 2. Only in school would a person have to provide the exact answer of $\frac{3}{8}$. Adults tend to use problem-solving techniques specific to a situation rather than the more generalized procedures taught in school. This use of a variety of methods may explain why studies fail to show a relationship between performance on arithmetic tests (computation, basic fact recall, word problems, etc.) and performance in solving many of the problems encountered in daily life.

In a revised mathematics program, more attention needs to be paid to ensuring that students understand the connection between problem solving in the real world and the arithmetic taught in school. Most importantly, we must move away from the overused practice of assigning calculation exercises, typically done in the absence of any practical context. We also should ensure that students learn and are permitted to use informal approaches to solving quantitative problems.

The Mathematics of Employment

Curriculum developers and teachers need to be aware of the relationship between the mathematics taught in school and the future needs of the student in employment. In the recent past, however, few attempts have been made to evaluate the adequacy of the mathematics program in preparing students to take up careers once they leave school. At the very least, teachers, parents, and students need to know whether or not the mathematics program meets the requirements of employment.

Obviously, it would be highly impractical and undesirable to use employment needs to develop the mathematics program. But society has a right to expect that graduating students can carry out the straightforward mathematical tasks seen in most non-specialized jobs without needing an excessive amount of additional training in on-the-job mathematics.

The Mathematics of Non-technical Careers.

The ACACE and the naturalistic studies discussed previously are primarily concerned with understanding how adults solve problems and carry out tasks seen outside the workplace. There are also many studies that have attempted to identify the mathematical demands inside the workplace by surveying employees, employers, supervisors, personnel officers and vocational teachers. Some results are too vague to guide decisions about curricular content. For example, one study (Morris & Cope, 1982) reported that Australian employers were looking for workers who had the following abilities or knowledge: "able to work with figures", "a bit of maths", "numbers", and "mathematics to a basic degree" (p. 30).

Other surveys usually attempt to elicit more detailed information from employers. In 1976, McIntosh asked 700 employers to indicate the importance of various mathematical topics listed on a questionnaire. The majority of the 100 employers who returned the questionnaire expected their employees to have the following competencies: compute with whole numbers and decimals; add and subtract money amounts; add and subtract fractions; calculate mentally; use tables; tell time; measure length and weight; and use SI units.

Richards (1978) interviewed employers from a variety of industries in London and identified the following basic mathematical requirements for these jobs: computing with whole numbers; understanding fractions and decimals; simple knowledge of ratios, proportions and percentages; the ability to read graphs, charts and tables; and facility with estimation and approximation. He also noted that the ability to use an electronic calculator was commonly reported as a desirable job skill.

In 1978, Swinson surveyed Australian employers in Queensland regarding the mathematics needed by students leaving school. He asked employers to rank a list of numeracy skills in order of importance for workers in their field. The majority of employers who returned the questionnaire - a dismal 145 out of 2500 - identified the following skills listed in their perceived order of importance as being very useful: simple addition, subtraction, multiplication and division of whole numbers; mental arithmetic; calculating with decimals; using a calculator; and working with percentages.

Closer to home, Smith (1978) attempted to identify the generic skills required by Canadian workers in various jobs. Over a period of several years, he surveyed workers in 49 occupations and supervisors in 28 occupations. Smith defined generic skills as "those job behaviours that are actively used in work performance, which are transferable from one job or occupation to another and which are needed for promotion to supervisory status" (p. 3).

Smith found that computation and measurement skills are common everyday requirements for most workers. Most importantly, he found that mathematics at work is always measurement related. In other words, computations such as adding $1/8$, $1/4$, and $1/2$ are **never** done without considering units of measurement. Workers are also expected to be accurate and verify their calculations and measurements. Charts and tables are used frequently by most workers. Increasingly, employers are providing their employees with calculators to promote productivity. Drawings and sketches are frequently used. Although the reading and interpreting of graphs is much in evidence in many occupations, few workers ever construct graphical displays.

Only in specialized occupations, such as engineering technology, is a knowledge of elementary algebraic and trigonometric notions needed. If algebra is needed in most occupations, it usually involves substituting numbers into formulas rather than manipulating algebraic expressions, the mainstay of many algebra teachers.

"Ready reckoners" are often used to eliminate the following kinds of complex calculations: looking up mortgage payments in a table; using a list to calculate sales tax; using a chart to determine antifreeze amounts. As Smith's study was completed before Canada began to change to the SI system, this type of knowledge was not studied. However, Smith predicted that this change would bring about a correspond-

ing decline in the use of fractions and a corresponding increase in the use of decimal fractions.²

While none of the studies of workers' mathematical requirements involved direct observation, there have been a few studies (Hiscox, 1980; Pirie, 1981) involving the observation of workers as they used mathematics in carrying out their on-the-job responsibilities. Hiscox interviewed both employers and employees and observed on-the-job performances in a variety of industries within the Nottingham / Derby area. These included construction, manufacturing, agriculture, printing, catering, the retail and distributive trades, and health and hospital work. She deliberately ignored the engineering industry because much earlier work had been done in this job sector. She also attempted to identify the type of training expected for each job and the selection tests used by the employers.

Pirie focussed on mathematics in medicine, especially the on-the-job mathematical demands of English nurses serving or training in a variety of health-related fields. The data for the report was collected in a variety of ways: visiting hospitals and observing performance in the wards; interviewing nurses in training, qualified practitioners and tutors; examining entry requirements, examination papers, and textbooks; analyzing over 550 questionnaires administered in hospitals throughout England.

An important observation was that those with a mathematical eye can describe many aspects of a job in mathematical terms; for example, stacking and packing crates can be described in terms of symmetry, translation and tessellations. However, if these mathematical ideas have been encountered in school, employees do not consciously use such notions to analyze the tasks they perform and, even if they did, their job performance would not necessarily be improved.

Mathematically-derived procedures are not necessarily the best way to perform a job. Workers can and do learn rule-of-thumb methods that are often more efficient than school-taught methods. Even trade schools can be faulted for ignoring the importance of on-the-job training. For example, Hiscox points out that a textbook used in a course for prospective sheetmetal workers discussed the installation of eavestroughing. It gave complicated formulas for the calculation of the bore and number of pipes needed for a roof of a given size and pitch, sometimes taking the local rainfall characteristics into account. She found that few installers use these ideas in actual practice. Instead the necessary sizes and numbers of pipes and troughing are decided from experience and, as is common in most trades, "err on the generous side when in doubt." This finding illustrates an important difference between the way mathematics is used in employment and the way it is often encountered in the classroom. The idiosyncratic nature of employment mathematics is a partial explanation for the criticism of school math voiced by employers in recent years.

The Mathematics of Technical Careers.

All industrialized countries, as well as those aspiring to become industrialized, place great stock in providing education for their masses. The current interest in education seen internationally can be largely attributed to a concern to produce highly skilled workers who can help their nation attain an advantage over other competitors in the global market. In both Canada and the U.S.A., there is grave concern that we are losing our ability to compete with Europe and the Pacific Rim in developing new products and new knowledge desired by others. There is also concern about the rising economic potential of both the Soviet Union and its Marxist neighbor, China - the most populated country in the world.

The technological advancement of a nation depends mainly on its ability to create and to draw from a large pool of highly trained and motivated scientists, engineers, and technicians. Obviously, it is not necessary for nations to rely entirely on their own resources to meet their personnel needs. Many countries, including Canada and the United States, meet their present needs by attracting highly skilled workers from other countries. However, for political as well as practical reasons, most countries prefer to employ workers educated in their own schools, universities, and technical schools rather than employ "brains drained" from other nations.

Canada and the U.S.A. are now entering an era where the public is acutely conscious of the economic benefits of locally-developed technologies and, as Hill (1982) suggests, "This consciousness is just one obvious step removed from a concern for the quality of mathematics education and science education as the bedrock of that foundation" (p. 108).³ Hill also summarizes the 1980 National Science Foundation and American Department of Education report Science and Engineering Education for the 1980s and Beyond that expressed a concern about the declining competitive position of the U.S.A: "This concern centers on the important question of whether the United States faces a reduced ability, relative to other countries, to generate and incorporate technological change in its production and utilization of goods and services" (p. 110). This report also expressed a concern about shortages in the engineering and computer professions:

While the United States has current shortages and future shortfalls in these areas, the Soviet Union, Germany, and Japan are producing much larger proportions of engineers and applied scientists than we are. At the same time, these countries are educating a substantial majority of their secondary school population to a point of considerable scientific and mathematical literacy, in part because they apparently believe that such literacy is important to their relative international positions. (p. 110)

The report recommends that the increasing shortage in the pool of technically trained personnel must be met in large part by producing more students who are both capable and interested in pursuing careers where advanced studies in mathematics, science, and other technical areas are required.

Similar concerns have been raised about Canada's ability to prepare technically and mathematically proficient workers. In a presentation to participants at the Joint Conference of the Association of Canadian Community Colleges and the Canadian Vocational Association in 1985, the vice-president of Gulf Canada Ltd., E. E. Walker, argued that in North America too much emphasis is placed on producing managers of economies rather on producing workers who can create new economies. He provided some interesting data to illustrate the differing attitudes of North American and Japanese industries. Out of every 10 000 people in Japan, only one is a lawyer and three are accountants. Out of every 10 000 citizens in the U.S.A., 20 are lawyers and 40 are accountants. Of the same group, Japan has 400 engineers and scientists, and the U.S.A has 70. Comparable figures for Canada break down roughly as follows: 14 lawyers, 62 accountants, and 156 scientists, engineers, and technical workers. Walker comments: "We have to ask ourselves if such a ratio between producers of paper and producers of goods represents the optimum development of financial and human resources in the competitive situation in which Canada finds itself." Clearly, Walker, as do many other influential industrialists, believes that more scientists, engineers and technical workers must be produced than ever before if Canada is to be able to compete economically with other nations.

But how can schools prepare mathematically capable and interested students for careers in science and technology while still meeting the needs of the vast majority of students who possess neither the ability nor the interest to pursue these mathematically demanding careers? Critics who believe that school mathematics programs can be improved simply by including more rigorous and more demanding mathematics suitable for those students preparing for university mathematics and science courses ignore the obvious fact that students vary greatly in their abilities to understand mathematical ideas, even relatively straightforward ones.⁴

In 1983, the Committee of Inquiry into the Teaching of Mathematics in Schools [hereafter referred to as the Cockcroft report] described a British study that found that there were a considerable number of 15- and 16-year-old students studying mathematics in secondary school whose understanding of place value was so poor that they could not write down the number 1 more than 6399; yet, there were 7- and 8-year-olds who could carry out this same task (p. 100). This study concluded there was roughly a 'seven year difference' in student's comprehension of mathematical ideas. Whereas an 'average' child can perform a mathematical task at the age of 11, but not at age 10, there are some 14-year-olds who cannot do it and some 7-year olds who can. The

'seven year difference' appears in studies of mathematical achievement in many other countries.

Even if students did not vary in mathematical aptitude and achievement or if new teaching methods were developed to overcome the 'seven year difference,' creating a mathematics curriculum based primarily on the needs of a relatively small minority - the needs of scientific and technical workers - would be a tragic mistake. The most optimistic predictions about future employment opportunities indicate that relatively few careers will require a knowledge of mathematics and science taught in the universities and technical colleges.

We need to be more realistic about individual differences in schools. By the time students reach secondary school, teachers seem to have three options to meet the mathematical needs of all students: 1) they can expect all students to adhere to uniformly rigorous standards; 2) they can teach for the average student; or 3) they can "water down" the mathematics to ensure that all students experience success. Teachers often find themselves in a "no-win" situation as the first and third options are politically unacceptable, and the second option leaves the talented and interested student bored and the less able student frustrated and disillusioned.

In summary, meeting the needs of scientific and technical workers is an important consideration in revising the mathematics program, especially at the secondary level. But the needs of the few should not be met by neglecting the needs of the vast majority of students who are neither interested in nor capable of studying mathematics at an advanced level. If our most capable students are to remain mathematically competitive with students in other countries, we need to ensure that they are provided with a mathematically challenging program presented at a rate commensurate with their abilities.

Summary of Employment Mathematics.

The on-the-job mathematical requirements of engineers, electronic technicians, retail workers, boilermakers, hairdressers, and painters vary considerably. Despite this great variation, it is possible to describe in general terms the types and levels of mathematics currently needed by the great majority of workers. While mathematics is not taught in schools solely for job preparation, the needs of the average worker should be kept in mind as new mathematics programs are developed.

Adding, subtracting, multiplying, and dividing with whole numbers are done frequently in all jobs, but rarely are large numbers involved. Calculations involving multiplying and dividing frequently involve one-digit factors and divisors: e.g., calculating the number of items in 5 boxes containing 24 in each. Workers rarely use pencil-and-paper methods to divide and multiply multi-digit numbers. Multiplying

and dividing larger numbers seems to be frequently needed only in occupations like engineering where similar calculations with decimal fractions are met.

Calculations involving decimal fractions to two decimal places are common in relation to money, especially addition and subtraction. The general use of decimal fractions has been increased by the use of SI units in Canada. Except for engineering, the mathematical practices of workers rarely involve the multiplication and division of decimal fractions. In fact, other than hypothetical situations posed in training institutions, Hiscox (1980) could not find one example of an on-the-job task where long multiplication or division of decimals was needed.

Aside from recognizing fractional parts involving halves, quarters, etc., few workers regularly use fractional arithmetic in their jobs, especially since the changeover to SI units of measurement. Only straightforward conversions between fractions and decimals are seen in employment: e.g., $0.50 = 1/2$, $0.25 = 1/4$. Technical workers have a greater need to use these types of equivalences.

Common fractions when used on the job almost always involve denominators that are powers of 2: fractions such as $3/16$, $5/8$, and so on. These types of fractions are seen mainly in occupations that still depend on Imperial units of measure. Calculations involving other types of fractions - adding $4/7$ and $3/5$, for example - would rarely if ever be seen in employment situations. Multiplication and division of fractions are seen even less frequently than addition and subtraction of fractions.

Hiscox found that, in the few cases where these operations were needed, standard practice was to carry out the calculation on a calculator by converting each fraction to a decimal. He identified the need to ensure that students are able to use a calculator to carry out some forms of fractional arithmetic.⁵ To calculate with fractions on a decimal calculator, students need to acquire the following knowledge: (1) know how to enter a fraction into a calculator (e.g., $3/4$ is entered as "3 divided by 4"; (2) know how to estimate an approximate equivalence between commonly seen fractions and their decimal counterparts (e.g., 0.55 is approximately equal to $1/2$); (3) be able to interpret the decimal display in terms of a common fraction (e.g., $3/16 + 1/32$ results in a display 0.21875 and can be interpreted as $22/100$, $1/5$ or even as the exact fraction $7/32$, depending on the degree of precision required).

To increase speed and accuracy and, ultimately, productivity, most employers encourage their workers to use calculators. However, because the majority of workers who use calculators regularly have not been trained in their use either at school or on the job, calculators are frequently not being used in the most effective manner. A concern of some employers is the failure of workers to notice errors in calculation, placing a decimal point incorrectly, for example. The most obvious implication for

change in the mathematics program is to place less emphasis on learning algorithms that are more efficiently done by machines and more emphasis on using calculator technology in a confident, efficient, and intelligent manner to handle the arithmetic normally needed to carry out practical tasks and to solve realistic problems.

Outside of the engineering and technical fields, few careers call for an extensive knowledge of the mathematics currently taught in secondary schools. The only aspect of algebra to be found by Hiscox involved straightforward substitutions of numbers in a given formula. Formula transpositions and equation solving are rarely seen or needed.

In conclusion, for the large majority of students who eventually enter the workforce, mathematics, especially that contained in the secondary program, cannot be considered essential preparation. Increasingly, educators are faced with a dilemma: how can a single mathematics program prepare the large majority of students for occupations that require little mathematics while simultaneously prepare a small minority for careers using a great deal of mathematics, perhaps much more than is present in the current secondary mathematics program? Clearly, it can't.

In revising the curriculum, the needs of employees, employers, and consumers need to be considered but not restrict the mathematics taught in school. Wheeler (1989) argues that from the point of view of society as a whole, all knowledge has an instrumental value:

Knowledge contributes to the life and health of that society; it is instrumental to its well-being. A healthy society will give considerable autonomy to its citizens to decide their own goals and conditions of life, so the knowledge that is of direct use to some is not to others. Theoretical knowledge, which has instrumental value for only a handful of people, may nevertheless open up a universe of experience rich enough to allow the participation of all members of society. Indeed, an education that remains always at the level of "only utility counts" is in a poor position to fulfill any goals, even utilitarian ones. (p. 284)

Footnotes

¹ Throughout this report, SI [metric] units will be used, except when referring to a report or article using Imperial units.

² Both Smith and Morris & Cope provide detailed tables of their results. Smith lists the generic mathematics and reasoning skills used in Canadian occupations. He also lists the core and non-core numeracy clusters. Morris and Cope list the specific literacy and numeracy demands on graduates entering selected apprenticeships in Queensland,

Australia. The results of the two studies are very similar. These lists can be used by the curriculum writers to compare employment requirements with the mathematics taught and learned in school.

- ³ This concern about the quality of math and science programs is certainly not new. When the Soviet Union launched Sputnik in 1957, the Americans saw this accomplishment as a threat to its heretofore unchallenged position as a world leader in scientific and technological advancement. The hue and cry raised by politicians and the general public led to considerable efforts, supported by both private and government funding, to reform the school mathematics and science programs. Although more recent and sober analyses suggest that only superficial changes were made in the 1960s and that many of these changes were misguided (see NACOME, for example), the world-wide technological dominance of the U.S.A over the 20 years following Sputnik led most people to believe that this massive and rapid effort at educational reform was successful.
- ⁴ A classic example is the Cambridge report of 1963 that proposed to change the mathematics program so that all students completing K-12 would have a "level of training comparable to three years of top-level college training today."
- ⁵ Recent advances in calculator technology may have already obsolesced this need. For example, the Texas Instrument TI Math Explorer has a built-in fraction algorithm that makes it easy to input, display, (e.g., "three quarters" is displayed as $3/4$) and manipulate fractions. The calculator will also perform fraction-to-decimal and decimal-to-fraction conversions.

CHAPTER 2

MATHEMATICAL PREPARATION OF INFORMED AND RESPONSIBLE CITIZENS

Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.

H.G. Wells

There are three kinds of lies: lies, damned lies, and statistics

Disraeli

Sixty-seven percent of the doctors surveyed preferred X to Y. (Jones couldn't be persuaded.)

Anonymous

Ninety percent of this game is half mental.

Every time he [Ray Fontenot, a pitcher] loses, he's got a better chance of winning his next time out.

Ninety percent of the putts that fall short of the cup don't go in.

Yogi Berra

In our concern to prepare students to use mathematics in a variety of possible careers and to carry out the routine tasks of daily living, the mathematics needed to be an informed and responsible citizen is often ignored. But the continual well-being of our democracy depends on the ability and willingness of its citizens to understand the quantitative information and arguments based on this information presented by governments, businesses, and other special interest groups. The graduates of our schools must leave with at least the knowledge, and we hope with the will, to reason statistically and to be familiar with the common misuses of statistics and probability.

Interpreting Statistical Information

Statistics - which refers broadly to the collection, analysis and interpretation of numerical data - is one of the important areas of mathematics utilized in our technological society, yet it receives only superficial treatment in current mathematics programs. All students must become familiar with rudimentary statistical notions and be encouraged to examine critically any statistics cited as evidence for an argument or used to express a point of view. Students need to learn when it is appropriate to collect statistical information, as well as the reasons for doing so. They also need to recognize the legitimate ways in which data can be manipulated mathematically and to appreciate the dangers of extrapolation or generalization. A revised program should aim to develop the qualities expressed in the Cockcroft (1983) report:

... a feel for numbers, an appreciation of appropriate levels of accuracy, the making of sensible estimates, a commonsense approach to the use of data in supporting an argument, the awareness of a variety of interpretations of figures, and a judicious understanding of widely used concepts such as means and percentages. All these are part of everyday living. Good statistical teaching can encourage pupils to think in these ways. (p. 236)

As a start in preparing an informed citizenry, students need to learn how to read and interpret quantitative information presented in any number of forms. In the early grades, students need to develop a facility with our base ten number system, especially an understanding of how symbols are used to represent quantities. The emphasis should be on handling numbers seen in everyday situations such as the following: reading prices; reading tabular data; reading and interpreting measurements; and reading and interpreting dials, calculator displays, and gauges.

Most children (and adults, for that matter) have only a vague "feeling" for the size of numbers commonly seen in newspapers, magazines, and information pamphlets.¹ In his imaginative book *Innumeracy*, John Paulos (1988) argues that few people have an intuitive feel for very small or very large numbers. He cites a study that showed that most doctors' assessments of the risks of various operations, procedures, and medications (even in their own specialties) were often in error, sometimes by several magnitudes. He provided the following personal experience as additional evidence:

I once had a conversation with a doctor who, within approximately twenty minutes, stated that a certain procedure he was contemplating (a) had a one-chance-in-a-million risk associated with it; (b) was 99 percent safe; and (c) usually went quite well. Given the fact that so many doctors seem to believe that there must be at least eleven people in the waiting room if they're to avoid being idle, I'm not surprised at this new evidence of their innumeracy. (p. 8)

More attention must be paid in school to the study of numbers as they appear in meaningful, everyday contexts. For example, government budget information such as GNP and deficits are often reported in millions, billions or even trillions of dollars, while pollution standards are often referred to as parts per million, or even parts per billion. These types of large and small numbers are not easy to fathom unless they are placed in more meaningful contexts as illustrated in the following examples:

A deficit of one billion dollars can be imagined as a line of one dollar bills laid end to end 4 times around the equator, an area of 4 sections of land covered with one dollar bills, or a per capita debt of \$1000 for every man, woman and child living in Saskatchewan.

A pollutant standard of 1 part per billion can be thought of as 1 drop of water placed in a body of water equal to five Olympic-sized swimming pools.

As a general principle, whenever any large or small quantities are presented in school, students should be encouraged to consider some reference to an everyday equivalence. Otherwise, arguments based on relatively meaningless quantitative information will be neither understood nor appreciated.

One important aspect of data analysis is the reading and interpretation of information organized in tables and charts or displayed as graphs. To develop a facility with reading these types of data presentations, it is important that a revised mathematics program contain extensive instruction on reading and interpreting data commonly seen in the world outside of school: telephone, electrical, and other utility rate tables; taxation tables; tables, charts, and graphs taken from Statistics Canada and other types of government publications; and polling and survey results.

It is most important to show students how the re-organization of data can be used to influence and, in some cases, deliberately mislead the unsuspecting or overly trusting reader: for example, changing the scale of one axis of a line graph so that relatively small increases are exaggerated and, consequently, a growth over a period of time appears to be more rapid than it actually is. Darrell Huff (1954) in his classic book *How to Lie with Statistics* provides many examples of how public opinion can be manipulated by carefully managing the way quantitative information is presented.

A revised mathematics program should provide students with regular opportunities to collect, organize, tabulate, and represent data through experimentation, by conducting surveys and by examining publications of statistical information. Elementary and middle grades' students can be asked to collect and tabulate data gathered on the TV viewing habits of children. Such information can be used to answer a question such as the following: "Do older children watch more TV than younger children?" This type of informal survey data can be compared to the broader surveys conducted by professional pollsters. As a general rule, however, students are usually more interested in analyzing the data they collect than in analyzing the data collected by others.

Little instructional time should be devoted to the construction of graphs, as people are rarely required to produce carefully drawn graphs. If suitable computer plotting programs are available, some instruction should be provided on their use. The graphs done manually by students should involve no more than the production of simple sketches. The program emphasis should be on analysing rather than on producing graphs and tables.

Politicians, businesspersons, bureaucrats, media people, and advertisers commonly assume that the people understand - or, more properly, hope they misunderstand - the many statements and arguments based on quantitative jargon. However, it is clear from the ACACE study discussed in Chapter 1 that terms like inflation are often widely misunderstood. Few people understand how inflation is measured and how changes in the rate of inflation can be interpreted: for example, a drop in inflation will not be followed necessarily by a drop in prices and an increase in the CPI does not necessarily need to be offset by a corresponding increase in salary in order to maintain the same standard of living. A revised mathematics program, should develop an understanding of the various indices and measures commonly reported in the media as well as their uses and abuses. These include gross national product (GNP), consumer price index (CPI), inflation, surplus, mill rate, deficit, unemployment rate, and so forth.

There is no need to include the more complex notions of hypothesis testing seen in most courses on statistics, although these notions should be introduced to the mathematically interested and capable students in secondary schools. The mathematics program should help all students develop a more critical attitude towards conclusions based on statistical information, especially when such conclusions are based on seemingly simple and straightforward statistics. For example, a Western Canadian politician was reported in a newspaper article as claiming that 86% of the welfare recipients completing a work-training program sponsored by the provincial government found permanent employment, a seemingly impressive statistic; however, it is based on the number of participants who completed and not on the number who began the program. The article reported that when the number of dropouts are included in the calculation less than a third of the workers who begin the program actually find full-time work.

Students need to learn that all special interest groups, including politicians of every stripe, are very selective when choosing statistical information to support their views. The study of mathematics education should help students develop a critical but not cynical attitude towards statistical information. As Weaver (1982) argues: "It is silly and irresponsible to uncritically either dismiss or accept arguments based on numbers; neutrality is no longer a tenable position; students need to be at least functionally literate with statistics" (p. 289).

Students and adults are far too trusting of statistical presentations for the continued health of our democratic society; too often, people substitute arithmetic for good judgement. Students need to be regularly exposed to real-world examples of statistical wizardry in order to counter the suspension of "common sense" often brought about by the magic and appeal of numbers.

Students should learn to analyze critically any studies and surveys whose findings are based on information that is difficult to quantify. In a study recently reported in the media, nearly a million women in Canada were estimated to be abused by their partners each year. This statistic compared favourably with a previous estimate that 1 in 10 women were abused. This sensational information has been considered controversial, primarily because battering is broadly defined to include emotional as well as physical abuse. For example, shouting at a partner or mistreating the family pet in the presence of a partner was counted as "abuse" in this study. Students need to learn that when important sociological phenomena are studied by researchers, quantification is not a straightforward process; much subjectivity usually underlies the supposedly objective statistics that are collected and later reported.

Sampling, and its use in drawing inferences about a population, should also be studied. Real-world examples can be drawn from many of the following areas: drug testing, diet effectiveness, wildlife population studies, pesticide testing, environmental studies, marketing studies, product testing, census surveys, opinion polls, and life-style assessments. Students should learn the importance of using a representative and sufficiently large sample. They should also learn to recognize instances of biased sampling. For example, almost all write-in and phone-in surveys conducted by newspapers, radio stations and TV stations are biased. This bias occurs for two reasons. First, the surveys are usually based on the opinions of readers, listeners, and viewers. The opinions of those who don't read that specific newspaper, listen to that radio station, or view that particular TV station are ignored. Second, the statistics gathered are based only on the views of people who are willing to participate in the survey. The views of the non-participants are ignored.²

Students should also learn to question conclusions based on an inadequately small sample, a commonly seen misuse of statistics. Although a great deal of experience and technical expertise are often needed to determine the adequacy of sample size, students can learn to identify samples that are obviously too small. For example, if 10 people are surveyed about their preference for two brands of soft drink and 6 choose brand A, it is obviously an over-generalization to claim that about 6 out of every 10 people prefer brand A.

New federal legislation requiring the media to reveal the details of polls, instead of only the results, provides students with many opportunities to work with statistically presented information. The media is required to present the following polling details: the size of the sample, the identity of those who commissioned the study, the results, the actual wording of any questions posed to the sample, and the sources of error. Much classroom discussion can centre on polling characteristics: for example, whether the wording of the questions may have influenced the way people responded and, consequently, biased the results.

Students also need to learn to be wary of claims and arguments based on incomplete information. This tactic of misrepresentation is often evident in commercials where advertisers try to demonstrate the virtues of their products. TV viewers are regularly exposed to the claim that "90% of dentists recommend toothpaste X to reduce tooth decay." The critical viewer knows that it is important to identify the missing as well as the given information. In the case of this commercial, the number of dentists surveyed has not been mentioned. Nor is the viewer told if other toothpaste brands were recommended by the dentists as well. Perhaps 90% of dentists recommend all common brands because there is no real difference in effectiveness among the brands.

A more subtle example of trying to blind the consumer with statistics is provided by Swift (1983). A television commercial touting the results of a marketing survey on two brands of razors stated that 90% of the people surveyed thought that brand A was either better than or no different from brand X. The viewers were not provided with information about the breakdown of this 90 percent. It is possible that 90% of the sample concluded there was no real difference between the two brands and the remaining 10% favoured brand X, not brand A. Swift reported that only 14% of grade 12 students in British Columbia recognized the weakness of the claim. (I suspect that adults would not fare any better than students in identifying the withheld information.)

The studies of mathematics in adult life discussed previously indicate that students and adults alike find percentages confusing, especially information expressed as a percentage increase or decrease. For example, most adults mistakenly assume that a discount of 25% followed by a further reduction of 25% results in a total reduction of 50%. When a base amount is successively increased or decreased by a percentage, as in compound interest calculations, even educated adults demonstrate a remarkable lack of understanding. A few years ago, a representative of a large Saskatchewan employer tried to convince a labour arbitrator of the generosity of the employer's wage offer by presenting him with evidence that the employees had enjoyed a substantial average annual wage increase over a period of several years. Much to the embarrassment of the employer, the employee's representatives pointed out that this statistic was all "smoke and mirrors" because it was calculated using compound rather than simple interest.

All students need to be introduced to common statistical terminology such as average, variation, percentile, and covariation. The focus of instruction should be on interpreting rather than on calculating. When calculation is necessary to help develop the understanding of a statistical notion, simple numbers should be used; when handling awkward real-world data, calculators and computers should be used.

Two measures of central tendency, the mean and the median, are seen almost daily, yet few people are aware of their differing properties. When analyzing data that is

distributed relatively symmetrically - for example, the human characteristics of height and weight, or clothing sizes - one kind of average is as good as another. However, in describing the average of skewed or asymmetrically distributed data - salaries, for example- the type of average used is important information because the mean is greatly influenced by extreme data. Students need to be provided with practical instances of the appropriate and inappropriate uses of these two measures.

Students need to be provided with at least a rudimentary understanding of covariation. In the secondary mathematics program, correlation techniques should be studied as an application of linear functions. Statistical calculators and statistical computer programs should be used to make the necessary calculations. Students can use the machine-derived answers in learning the following statistical ideas: interpreting negative and positive indices predicting the direction and estimating the size of a correlation index suggested by a rough scattergram of data, and predicting the effect on the index when numerical transformations of the data are performed (e.g., knowing that adding a constant to one or both measures does not change the size of a correlation). The more able students should be exposed to other types of correlation measures.

The study of covariation should not be overly technical, however. It should not be assumed that students who can perform the mechanical aspects of statistics can necessarily use statistical ideas intelligently. The focus of attention must be primarily on the application and interpretation of covariation. Students must learn that the existence of a correlation does not necessarily suggest that a cause-effect relationship exists. They should also learn to identify and explain spurious relationships. For example, I suspect that there is a high positive correlation between the height and weight of a teacher and the grade level they teach. Why? Simply because more males than females teach in the secondary grades and males are generally taller and heavier than females.

Spurious relationships are reported regularly in the media. A graph included in a large advertisement placed in a Saskatchewan newspaper showed that as funding increased for Planned Parenthood programs the number of therapeutic abortions performed in Saskatchewan increased and, conversely, a decrease in abortions followed a decrease in funding. The obvious conclusion left to the naive reader is that somehow Planned Parenthood programs are the sole cause in a rising abortion rate. Other alternative explanations for the rise and fall in the abortion rate are simply not explored. If students are to learn to think critically, the questionable statistical methods of all advocacy groups must be exposed.

Interpreting Chances and Risks

Adults are exposed regularly to statements of probability and, as mathematics educators, we should ensure that students understand how to interpret these statements. Applications such as the following should be discussed in school: predicting genetic characteristics, calculating drug efficacy, estimating contraceptive failure, assessing health risks and recovery rates from illnesses, estimating risks from pollutants and nuclear reactors; developing life insurance mortality tables, forecasting weather trends; and determining the chances of winning and losing lotteries and other games of chance.

The basic terms and definitions seen in the study of probability - such as event, outcome, chance, odds, frequency, combination, likelihood - can be introduced in the middle grades' mathematics program, after the students have acquired a reasonable understanding of fractions. The instructional approach should be activity-based as much as possible and incorporate simple experiments relating empirical and theoretical probability. Students should collect, organize, tabulate, graph and interpret information based on experiments involving dice throwing, coin flipping, thumbtack tossing, or card drawing. The results of these experiments should be compared to the theoretical probabilities calculated by compiling an organized list of all possible outcomes such as listing all pairs of outcomes possible when two coins are flipped simultaneously.

Once the possible outcomes for various conditions have been enumerated, the following types of questions about theoretical probabilities can be considered: (1) What is the probability that both coins will read heads? (2) What is the probability that a head appears on neither coin? (3) What is the chance that at least one head appears on a coin? (4) About how many times would both tails and heads appear if two coins are tossed 10 times, 100 times, and 1000 times? For experiments requiring a large number of trials, a programmed simulation conducted on a computer is far more practical than actual experimentation. The lack of agreement between the expected frequencies and the observed frequencies resulting from experimentation can serve as an illustration of the law of large numbers - smaller samples are more variable than larger samples.

In the secondary mathematics program, more advanced notions of probability should be introduced, although a great deal of technical expertise is not really necessary to understand the most common applications of probability. For example, few people have either the knowledge or the interest to calculate the chances of winning Lotto 649, but most can learn to interpret intelligently the expressions of probability associated with lotteries. Schools should ensure that all students are "lottery literate."

Opinion polls are another common use of probability that can be introduced into the mathematics classroom. Few people appear to understand how to interpret the margin of error, often expressed as a probability. For example, prior to the last federal election a newspaper reported that a Gallup survey found that 32%, 31%, and 29% of decided voters supported the Liberals, NDP, and Conservatives, respectively. Despite carrying the statement that the poll was considered accurate within four percentage points 19 out of 20 times, (and, therefore, no real difference in popularity among the parties existed) the newspaper claimed that the Liberals were leading the political pack while the governing Conservatives were running dead last.

Another application of probability for the mathematics classroom involves the subject of risks, especially decision-making based on the probability that some situation may or may not arise. The calculation of risk probabilities differs significantly from those usually presented in mathematics textbooks. At best, they are based on measurement and observation rather than being based on mathematical analysis. For example, the probability of being involved in an automobile accident or acquiring AIDS is based on statistics about traffic accidents and the incidence of AIDS. At worst, risk calculations are simply fancy guesswork. Students need to learn to recognize situations where probabilities are calculated from theory rather than on actual measurements collected over time.

A knowledge about probabilities does not guarantee that people will necessarily make the best decisions. Most smokers understand the health risks associated with smoking, yet they continue to smoke. Just as price is not the only consideration in making a purchase, probability alone does not dictate decisions in the matter of taking risks. However, schools must ensure that when its graduates accept risks, they do so not out of ignorance of mathematical probabilities but because of reasoned choice.

Laplace, the famous 19th century mathematician who put the study of probability on firm mathematical ground, claimed that the theory of probability is only common sense expressed in mathematical language. Unfortunately, common sense regarding some notions is not all that common and there many aspects that students and adults find especially confusing (see Hope & Kelly, 1983, for a discussion). A revised mathematics program should anticipate these difficulties in probabilistic reasoning and include instruction to help students overcome their difficulties.

People are generally perplexed by the notion of uncertainty. Students need to learn that an event estimated to have a low probability will rarely occur and, conversely, an event estimated to have a high probability will likely occur. But they must also understand that even highly improbable events can and do take place - given the right circumstances, sufficient time and enough instances - and that even probable events might not occur. Unlike most of the mathematical notions studied in school, probability can be discomfoting because there are often no hard and fast rules to guide

one's decision making. The "gamblers' fallacy," the belief that random events are dependent, is another commonly misunderstood idea about probability. The 1977 U.S. National Mathematics Assessment (NAEP) found that the majority of 17-year-old students suffer from this form of faulty reasoning. Over 70% responded incorrectly to this item: "In three tosses of a fair coin, heads turned up twice and tails turned up once. What is the probability that heads will turn up on the fourth toss?" (p. 319). Only 30% realized that the probability remained at one half, regardless of the past performances of the coin. Students need to be aware that random events do not have memories for past events.

There are many real-world examples that illustrate how tempting it can be for people to succumb to the gambler's fallacy. For example, when a woman gives birth to a consecutive number of babies of one sex, it is commonly assumed that there is an increased probability of having a child of the opposite sex. Perhaps Mrs. Emory Harrison, who was reported as giving birth to thirteen consecutive boys, was inspired to further pregnancies by a mistaken belief that the next child "just has to be a girl."

Many people also seem to abandon their common sense when they purchase lottery tickets. They do not seem to understand that fairly-run lotteries use a random process to select the winning numbers and, consequently, all ticket numbers have an equal chance of being chosen. Yet many people will not choose certain number combinations such as 123456, 555555, or 121212 because they mistakenly assume that these ticket numbers are so "remarkable" that they are less likely to be winners. The wisdom of avoiding remarkable numbers is reinforced when it is noted that no winning lottery numbers are particularly remarkable. The fact that less remarkable numbers did not win either is largely ignored or goes unnoticed.

Students need to learn to recognize the questionable uses of probability, especially those schemes that border on the fraudulent. Recently a newspaper ran a full page ad in which a supposed "numbers genius" had developed a successful scheme for winning lotteries. For the meager sum of \$20.00, she was willing to share her knowledge with others. The ad contained a number of statements that could be used in the mathematics classroom to help students think critically. For example, the ad made a remarkable claim that the scheme "increased one's chances of winning Canadian or U.S. lotteries by more than 500%." Students should realize that anyone's chances of winning will be increased by 500% simply by purchasing 5 tickets instead of one.

As informed citizens, students need to understand how mystics, astrologers and other charlatans use the misunderstood notions of probability to prey on the gullible and ignorant. Paulos states, "Innumeracy and pseudoscience are often associated, in part because of the ease with which mathematical certainty can be invoked to bludgeon the innumerate into a dumb acquiescence" (p. 49). Astrological predictions are often

based on the rarity of some heavenly event such as a particular alignment or arrangement of planets. These types of predictions ignore the fact that one heavenly arrangement is as likely as any other arrangement.

Coincidence is another misunderstood notion that needs to be dealt with in school. I recently viewed a popular American documentary program on T.V. that reported the remarkable story of a man who dreamt of a murder that happened the next day. This coincidence can be explained, as can so many other coincidences, by considering plausible events: (1) Given the considerable publicity given to violent acts, especially murders, it is highly probable that a large number of Americans dream about murders; (2) Many of these dreams will be similar to actual murders; (3) A few dreams will be so similar to actual murders that they will be given an unusual amount of publicity. Students need to understand that in large populations it is almost certain that someone at sometime will experience a "remarkable coincidental experience."

The current mathematics program falls far short of the goal of producing citizens who are numerate enough to understand the uses and abuses of mathematics and who can analyze claims and evidence based on quantitative information. In 1986, a Gallup poll found that 52 percent of American teenagers subscribe to one tenet of astrology: our lives are controlled by zodiac influences. I suspect that teenagers in Saskatchewan are equally gullible. Certainly, the mathematics program cannot be blamed entirely for student belief in pseudosciences. However, the current secondary program may unwittingly contribute to the "number numbness" by not emphasising the applications of mathematics, especially the use of statistics and probability to explain everyday events.

If the development of numeracy continues to be considered less important than the acquisition of techniques and the accumulation of terminology that are soon forgotten, students will continue to be bamboozled by mathematical trickery. The mathematics program needs to be revised to ensure that students can function in a society where quantitative information is increasingly used to dupe the innumerate. Teachers will need to be provided with practical teaching materials to help students understand the real-world uses of data analysis, statistics, and probability.

Footnotes

¹ A poll conducted by Angus Reid and Associates in March, 1989 for the National Citizens' Coalition found that only 47% of Canadian adults 18 years of age and over know that one billion dollars is equivalent to 1 000 million dollars. Of those surveyed, 6 percent thought the answer was 10 million and 28 percent said 100 million. The president of a Canadian political party in praising the latest federal budget equated a budget saving of \$3 billion to ten thousand people earning \$30 000 annually - he erred by a factor of 10.

When the error was drawn to the attention of a party representative, she said "We really did an injustice by losing the extra zero. The number is so big it went off the calculator."

- ² A radio station in Saskatoon posed this question: "Should the University of Regina be closed and the savings be used to support the University of Saskatchewan?" Not surprisingly, the majority of listeners agreed with the statement. I voted "yes," three times.

CHAPTER 3

MATHEMATICS AND LIBERAL EDUCATION

From the beginning of its introduction into formal schooling, the study of mathematics has contributed significantly to the development of a general or liberal education, the cultivation of a student's intellect through involvement with the best intellectual achievements of scholars over the centuries. In becoming liberally educated, a person acquires knowledge about several important fields of human endeavour including understanding their historical and cultural merits, and learns how to think rationally and critically. In a liberal education, there is less concern for "practical matters."

The acquisition of knowledge is a desirable goal of a liberal education. But the development of less tangible qualities such as logical thinking, sound work habits, a respect for accuracy and truth, and an appreciation of aesthetic matters are considered equally important goals. Mathematics educators often make exaggerated claims about the contribution of mathematics to the development of these qualities.

Logical Thinking

For centuries, mathematics has been considered a useful medium for training people to think logically and accurately and to acquire a respect for truth and precision. For La Chalotais, in his *Plan d'Education Nationale* of 1763, it was "very possible and very common to reason badly in theology, or in politics; it is impossible in arithmetic and in geometry; if accuracy of mind is lacking, the rules will supply accuracy and intelligence for those who follow them." For Diderot, geometry was the "best and simplest of all logics, and it is the most suitable for fortifying the judgement and reason." Jeremiah Day, a president of Yale and respected academic, wrote in the preface of his introductory algebra book published in 1823 (Travers, Pikaart, Suydam, & Runion, 1977)

If the design of studying the mathematics were merely to obtain such a knowledge of the practical parts, as is required for transacting business, it might be sufficient to commit to memory some of the principal rules, and to make the operations familiar, by attending to the examples... But a higher object is proposed, in the case of those who are acquiring a liberal education. The main design should be to call into exercise, to discipline, to invigorate the powers of the mind. It is the logic of the mathematics which constitutes their principal value, as a part of collegiate instruction. The time and attention devoted to them, is for the

purpose of forming **sound reasoners**, rather than expert mathematicians. (p. 7)

Day's view reflected the psychology of that area: faculty psychology or the theory of mental discipline. The theory essentially viewed the mind as a muscle whose reasoning processes could be strengthened through struggling with difficult ideas and through repetition and practice. One textbook writer in the late 19th century described the theory in this manner (Jones & Coxford, 1970):

The mind is cultivated by the activities of its faculties Mental exercise is thus the law of mental development. As muscle grows strong by use, so any faculty of the mind is developed by its use and exercise. An inactive mind, like an unused muscle, becomes weak and unskillful. Hang the arm in a sling and the muscle becomes flabby and loses its vigor, and skill; let the mind remain inactive, and it acquires a mental flabbiness, that unfits it for any severe or prolonged activity. An idle mind loses its tone and strength, like an unused muscle; the mental powers go to rust through idleness and inaction. (p. 27)

Although Thorndike's research disposed of this theory in the 1920s, many influential people still hold a view that appears to be strikingly similar. The following quote is taken from the Cambridge Report of 1963 attended by many eminent mathematicians, scientists and engineers:

Many poor patterns of thought in ordinary life may be modified by the study of mathematics. The misleading, but almost universal, assumption that things must be ordered linearly might be dispelled by the study of partial orderings. Just a little experience with logic and inference can do away with some of the unfortunate reasoning we meet all too often.

Recent surveys indicate that strongly held views about the role of mathematics in promoting logical thinking rarely disappear, even in light of contrary evidence. Smith (1989) reports a study that surveyed university professors and professional staffs on several American college campuses. He asked the respondents to indicate their agreement to the statement: "The study of mathematics is especially important in helping students learn to think logically." Sixty-two percent of all professors in the natural, biological, and physical sciences agreed with the statement, while only 18% disagreed. The remainder were undecided (p. 451).

Smith reports an exactly opposite trend among faculty members in other disciplines and among members of professional staffs: Only 26% of professors in the humanities agreed; only 32% of those in the social sciences agreed; only 40% of members of the professional staffs agreed. The researcher who conducted the study is reported by

Smith to have concluded, "The fact that so many of the 'hard' scientists believe in such a fundamental issue in the absence of the kinds of data they themselves would accept may be the presumptive evidence against automatic subject matter transfer."

Teachers do an excellent job in perpetuating this view. Robitaille, O'Shea and Dirks (1982) asked grade 12 students and their teachers to indicate their agreement with the following statement: "Mathematics helps one to think logically." Both students and teachers highly agreed with the statement. Fey (1981) found that among teachers the mental discipline theory is a widely held view. He reports the following typical statements of teachers when asked to find virtue in the study of mathematics: "I can teach them [students] to think logically about real problems in their lives today"; "Mathematics can teach the student how to think logically and that process can carry over to anything. To be able to start with a set of facts and reason through to a conclusion is a powerful skill to have" (p. 23).

One school board in Saskatchewan conducted a study to discover and report examples of critical and creative thinking in the classroom. In their zeal to identify some instance of critical thinking being promoted in the mathematics classroom, the authors wrote, "Mathematics courses at Collegiate X frequently demanded the application of critical thinking skills. In Mr. R's grade 9 Algebra class students were asked to interpret word problems and then translate them into mathematical symbols so that they could be solved. Mr. D's Geo-Trig 30 students were often asked to translate a problem into a diagram." Clearly almost any form of thinking can be considered "critical thinking," if we use our imaginations.

Although the study of mathematics is considered to somehow improve one's general thinking abilities, little empirical evidence exists to support this view. The reason that mathematics is viewed as an aid to logical thinking stems from a belief that mathematical deduction will benefit deduction in other areas. In other words, people will become more "logical" through their work in mathematics. However, people who are accused of being "illogical" usually do not have difficulty in understanding and using logical conventions such as a syllogism. The most common difficulty that people experience when reasoning is that their arguments and claims are often based either on false antecedents or on antecedents whose veracity is difficult to determine.¹

Powerful forces usually unseen in mathematics courses influence our reasoning outside the classroom. For example, the psychological context can radically affect our rationality. Kahneman and Tversky (1982) presented subjects with the following personality sketch:

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of

discrimination and social justice, and also participated in anti-nuclear demonstrations.

Respondents were then asked which of the following statements about Linda were most probable: (1) Linda is a bank teller; (2) Linda is a bank teller who is active in the feminist movement. It was found that 86% of undergraduate students and 50% of psychology graduate students judged the second statement to be more probable. Cobb (1986) in analyzing this result argues that the students' faulty reasoning was not necessarily due to a lack of understanding of elementary probability but, rather, it reflected the context within which they interpreted the term "probable." He concludes that what is regarded as logical problem solving in academic settings may not necessarily fit with problem solving in everyday situations, not because people are "illogical" but because practical problem solving requires efficiency rather than a full and systematic consideration of alternatives (p. 3).

In logic and in mathematics, antecedents are normally either true or false; in everyday life, we're often unsure. Consequently, it is misleading to state unequivocally that mathematics should be taught because it improves one's thinking capabilities. Moreover, even if mathematics does improve one's thinking in some minor way, it may not necessarily be the best subject to accomplish this goal. Perhaps, legal education, computer programming, diagnostic auto mechanics, or chess can do the job just as well.

Mathematics does involve logical thinking and, for this reason alone, some mathematics should be studied by all people. But rather than grandly claiming that mathematics can produce students who are sound thinkers, mathematics educators should be content with more modest claims and argue that learning to think logically while studying mathematics helps only in learning to think logically while studying more mathematics.

Culture and Aesthetics

The study of mathematics is often justified because of its cultural and aesthetic merits. G. A. Hardy (1940) frequently argued that mathematics should be studied for its own sake, primarily because of its beauty and elegance: "The mathematicians' patterns, like the painters' or the poets' must be beautiful; the ideas, like the colours or words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics" (p. 24).

In *Everybody Counts* a recent publication of the Mathematical Sciences Education Board (1989), the various merits of mathematics are identified:

Cultural - the role of mathematics as a major intellectual tradition, as a subject appreciated as much for its **beauty** as its power. The enduring qualities of such abstract concepts as symmetry, proof and change have been developed through 3,000 years of intellectual effort. They are to be understood best as part of the legacy of human culture which we must pass on to future generations. Indeed, it is only when mathematics is viewed as part of the human quest that lay persons can appreciate the esoteric research of twentieth-century mathematics. Like language, religion, and music, mathematics is a universal part of human culture. (p. 33)

Do students view mathematics as a beautiful and appealing subject? Do they appreciate that mathematics is an important field that is often studied for its own sake? Do they understand its history, and the way it was developed? Cockcroft (1983) reports that most people see mathematics in vocational utilitarian terms and, most importantly, very few see mathematics "as serving any wider purpose, especially of what might be termed a cultural or general educational kind" (p. 61).

Few teachers have the time in an overcrowded curriculum to help students appreciate mathematics for its beauty, its patterns, its elegance, its foundations, and its history. Such intangible aims require a major revision in the current program that emphasizes skill acquisition. However, most teachers lack the necessary background to develop these appreciations. For example, few mathematics teachers including secondary teachers have studied the history of mathematics and its impact on society. As *Everybody Counts* asserts, "... it is only through history that teachers will come to know that mathematics changes and to see the differences between contemporary and ancient mathematics" (p. 64).

In summary, mathematics has no special place in the development of critical thinking and aesthetic appreciation. The study of mathematics certainly can make some contribution to these ends but the extent to which it does so depends entirely on how and what is taught. These realities are often forgotten when curricula are developed.

Footnotes

- ¹ In a classic study of everyday reasoning done in the 1950s, white males living in the southern American states were able to select the correct conclusions of syllogisms based on 'neutral' antecedents: e.g. "All postmen wear running shoes. If Bill is a postman, does he wear running shoes?" But when the antecedent conflicted with their beliefs, they could not accept the premise and their reasoning suddenly became 'illogical': "All Blacks are intelligent and hard working. If Bill is a Black, is he intelligent?"

that are rarely used is largely a waste of time, especially when inexpensive hand-held calculators are so readily available in the world outside of school. For example, the time needed to achieve and maintain an adequate level of performance in long division has been estimated to take as long as two years of arithmetic instruction (Wheatley, 1980). Put another way, the cost of producing unreliable "human dividers," who rarely need to employ pencil-and-paper division outside of school, is approximately \$750 - \$1000 per student.

The time taken in school in learning how to compute often has been gained at the expense of learning when to compute. Many studies have found there is a considerable gap between a student's ability to compute with straightforward numbers and the ability to apply this knowledge to solve realistic problems. Producing students who are able to compute, but not able to solve problems, seems analogous to developing students who can read musical notation but can't use this knowledge to play a musical instrument. This point is made by Usiskin (1978):

The crucial issue is not how the calculation is accomplished but rather knowing when and how to use arithmetic to solve problems or answer questions that in fact matter in the lives of people. Insisting that all children must be excellent pencil-and-paper calculators puts the emphasis in the wrong place - on the means, rather than on the ends, of calculation. (p. 413)

Although not a replacement for thinking and reasoning, the hand-held calculator can provide teachers with additional time to help children better understand mathematical concepts that they traditionally find difficult to fathom. As a general rule, calculators should be used in school whenever a calculation by hand either takes up an unwarranted amount of time or obscures the mathematical process under study.

It will not be easy to usurp written computation from its current regal position. As the reformists in Benjamin's *Saber Tooth Curriculum* found, the comfortable ideas of the past are reluctantly abandoned. Many parents, teachers, and administrators view calculators as a threat rather than as an aid to numeracy, destroying the human intellect and undermining our native faculties of memory.¹ For example, one study (Wyatt, 1979) found that almost half of the teachers they surveyed thought that pocket calculators would diminish the memory and mathematical ability of children.

Others oppose calculator use on the grounds that those who avoid tedious calculations must suffer from a character deficiency. Hiscox (1980) noted what she thought was an unrealistic view of calculator use held by a bank manager she interviewed: "I had to do much more tedious things than this," he exclaimed. Hiscox comments that the bank manager "seemed to think that this type of arithmetic was somehow character building - a necessary attainment for any decent citizen" (p. 149). She

CHAPTER 4

TECHNOLOGY AND MATHEMATICS

Ever since the study of mathematics became a compulsory school subject, the major goal of instruction in the elementary and middle years mathematics programs has been the development of computational facility. For many people, being numerate means being able to use a pencil and paper to carry out standardized, Hindu-Arabic, place-holding algorithms for determining sums, differences, products, and quotients. But with the advent of calculators and computers, the elementary and middle years mathematics program with its emphasis on producing "human computers" is currently more suited for preparing shopkeepers in the 1940s than it is for preparing students who will graduate in the 21st century. This chapter will briefly review the implications of the hand-held calculator and the computer on curriculum and instruction.

Calculators and the Curriculum

Teachers continue to spend a considerable amount of time and energy in trying to ensure that their students master pencil-and-paper algorithms. One study estimated that approximately 90% of the instructional time in arithmetic is spent in learning or practicing computation. Despite the efforts of teachers, the development of computational facility seems an elusive goal, as indicated by the findings of recent provincial studies in British Columbia (Robitaille, 1981; Robitaille & O'Shea, 1985; Robitaille & Sherrill, 1977) and in Saskatchewan (Beamer, 1976; Giesbrecht, 1977; Randhawa, 1986). For example, Randhawa found that only about 20% of grade seven students in Saskatchewan could multiply and divide decimals accurately (p. 26). The grade ten sample showed little improvement in handling the more complex algorithms of rational number arithmetic (p. 31). Studies undertaken in several countries show that a large number of students and adults never become proficient in handling decimal and fraction computation. This lack of facility is not a recent phenomenon; the evidence of earlier studies suggest that a "golden age of calculation" has never existed.

Moreover, those unfortunate students who do not demonstrate a high level of computational proficiency when they enter high school are often denied access to the study of interesting mathematics. Instead, they are usually doomed to study a program of dreadfully dull "general" or "remedial" mathematics, where repeated efforts are made to teach computational skills not mastered in the earlier grades. But, as the literature review of off-the-job and on-the-job arithmetic discussed in Chapter 1 clearly indicates, continuing to place a heavy emphasis on the teaching of algorithms

thought it ironic that this bank was currently running advertisements for staff which suggested that little mathematics was now required.

As Levinson (1986) points out, this concern about the deleterious effects of technology on the human mind is not new. Socrates, in the *Phaedrus*, warns that writing will result in the atrophy of memory and the end of learning through dialogue. Fortunately for Socrates, his pupil Plato, troubled to write this warning down (p. 37).

Socrates, of course, was not incorrect in the immediate sense; access to written or any recorded material will diminish some human abilities. Likewise, calculators will probably lead to a general loss of ability to perform traditionally taught algorithms such as complex multiplication and division. (How many people can remember the traditional "doubling and squaring" algorithm for figuring square roots?)

But the enormous gains in mathematical power seem well worth risking a possible loss in the ability of students to produce a calculation manually. Although many studies demonstrate that the use of calculators does not hinder the computational facility of students (see the extensive review by Hembree & Dessart, 1986), this type of research seems hopelessly outdated and, for the most part, irrelevant. Why should we be concerned with a diminished ability to perform skills that are rarely used? Does anyone mourn the passing of Roman numerals?

The time has arrived for schools to take advantage of calculators - the greatest advance in portable computational aids since the Hindu-Arabic algorithms obsolesced the decimal abacus. We need to re-examine and revise some of the presently unrealistic expectations for computational mastery: for example, according to the Saskatchewan Division I and II curriculum guide, year 4 students are currently expected to be able to "subtract two numbers with minuends less than 1 000 000 with and without regrouping" and "multiply whole numbers involving a 3-digit factor by a 2-digit factor with and without regrouping." Although few teachers would assign such excessively large computations, unrealistic standards such as these continue to provide some teachers with the justification to continue to teach the computation they always have taught. There has been little change in these standards for many years; similar standards can be found in textbooks used 75 years ago in Saskatchewan.

Calculator use does not mean that all types of calculating methods are obsolesced. Calculators should be considered primarily as a problem-solving tool and treated in the same manner as a dictionary. Dictionaries are used by literate people to check the spelling, pronunciation, usage, origin and meaning of words. Just as literate people do not rely entirely on a dictionary to carry out their daily communication tasks, numerate people do not rely entirely on a calculator to carry out their daily computational tasks. For straightforward computations, the figuring is done either with paper and pencil or in the head. A revised mathematics program should ensure that

calculators are not needed by students to carry out such tasks as figuring a sum of 15 dimes, calculating the area of a 5 metre by 7 metre room, or figuring the number of eggs in a carton of 2 dozen. We need to ensure that students learn to distinguish between the appropriate and inappropriate uses of calculators.

Most high school mathematics teachers have incorporated the scientific calculator into the current program. However, I suspect that a substantial number of traditionalists continue to force students to work with logarithmic and trigonometric tables. Robitaille, O'Shea, and Dirks (1982) found that, although virtually all Algebra 12 teachers in British Columbia permitted the use of calculators by students, about half of them continued to emphasise the computational applications of logarithms in their teaching.

Computers and the Curriculum

High school teachers view calculators and various forms of calculating software as instructional tools that, for the most part, will have only a marginal impact on the mathematics curriculum. But this view will have to change because symbolic mathematical systems can be expected to have a significant effect on the high school and, perhaps the university, mathematics program. Symbolic manipulators such as muMATH have the capacity to perform all the standard processes currently taught in secondary school including the following:

1. polynomial algebra - addition, subtraction, multiplication, and division of polynomials.
2. trigonometric manipulations including the production of identities.
3. factoring and equation solving.
4. most standard calculus procedures.

When a considerable amount of high school mathematics is finally imbedded in a single silicon chip and portable symbolic manipulators are readily available to students, the secondary program will have to change significantly. The emphasis will change from memorizing a large number of symbolic manipulation procedures such as factoring to learning the properties and applications of these procedures.

Even instruction in geometry will have to change.² Several relatively new software packages are now available to assist in the teaching of geometry. The first type of program could be called an electronic compass. In this type of package the user identifies points on the screen with letters A, B, C, and so forth, and selects from a

menu of possible constructions: for example, construct the bisector of angle BCA , or draw a line perpendicular to BD at point E . Upon completing the construction of the figure, a wide variety of measurements is available including the length of sides, measures of angles, and the areas of triangles. If desired, these measures can be compared by division so that the ratio of line segments can be determined. Some packages include the ability to remember the set of instructions and repeat them step by step on other figures chosen by the user.

These programs allow the user to make accurate drawings, request information, and draw conclusions based upon the answers obtained. In one example of the use of these packages cited by a vendor, a student is given a triangle ABC with a mid-line DE drawn in. Key questions such as the following can be asked: (1) What is a midline? (2) What else appears to be true about a midline? (3) What are the ratios of the corresponding sides of triangle CED and triangle ABC ? (4) What are the ratios of the corresponding angles of these triangles? (5) How do the areas of these two triangles compare? (6) Which of your answers to the preceding questions always appear to be true? (7) How many of your assertions can you prove?

These electronic compasses help the student notice geometrical patterns and relationships, to hypothesize, and to collect data to support or refute a hypothesis. Geometry will not be a guided tour of theorem after theorem after theorem. It has the potential to become a process of reflecting, hypothesizing, and a search for proof rather than an accumulation of terminology and quickly forgotten proofs. The currently available list of packages which provide the user with electronic compass capabilities include the Geometric Supposer series from Sunburst and the GeoDraw software included in the IBM Geometry Series, Geometry One: Foundations.

A second type of package that will influence the development of the geometry instruction is proof checking or theorem proving software. In this package, the student states or reads a theorem and determines the "given" and the "to prove" portions in a rather conventional manner. The figure is drawn with the aid of GeoDraw, and an almost conventional 3 column proof is set up. There are columns in which statements and reasons are to be entered. The third column (actually, the middle column of the screen) is a "reference" or "support" column which identifies the steps of the proof where the hypotheses of the theorem being cited are satisfied. For example, if the user wishes to assert that two triangles are congruent by SAS, 3 lines would have to be referenced: one stating that a pair of included angles are equal (or congruent) and two declaring that the included sides are equal (or congruent). The computer system checks to determine that the hypotheses of the theorem are satisfied before it accepts the conclusion as a new truth for future use. When the user thinks that the theorem is proved, the word END is entered. After comparing the results with the "to prove" statement, the system will accept or reject the proof. If desired, a printed copy of the

computer generated proof can be produced for submission to the teacher. Furthermore, logically correct but unnecessary steps in the proof can be identified and even omitted. However, the teacher will still know that these steps were present because the steps in the proof are not numbered consecutively. For example, the steps may be identified as 1, 2, 4, 5, 8, and 9. Only statements which are logically justifiable will be accepted by the proof checking package. The proofs may be any correct approach (so long as the software has the support theorems in its memory) and the steps may be entered in any logically correct order. (The package described is the ProofChecker software in the IBM Geometry Two: Proofs and Extensions (with ProofChecker) package.)

In redesigning the mathematics curriculum, the use of technology must be carefully considered. All students must have access to simple four function calculators and, when students are ready, to a scientific calculator. Any traditional procedures that can be done faster and more efficiently by a calculator should be either de-emphasised or eliminated.

For several reasons, we will have to proceed cautiously in changing curriculum and instruction to accommodate computing software and symbolic manipulators. First, unlike the hand-held calculator, computers are still not sufficiently abundant in schools and in the home to ensure that all students have easy access to them. Second, major changes in support materials such as textbooks need to occur. They have to be designed to incorporate the new technologies in their design. Third, teachers, especially secondary teachers, are not yet convinced that computers have seriously diminished the need to learn the various mathematical routines and procedures they've been teaching for years. A study by Becker cited by Fey (1989) found that only about one third of mathematics teachers use computers in any instructional capacity and, when they do, almost 60% of the computer time is spent using tutorials or drills on the skills supposedly obsolesced by computers.

Our present situation is similar to a teacher in the 1960s being informed about the inexpensive technology to be made available in the 1990s. Would this teacher suddenly ignore the teaching of calculation knowing that it would soon become unnecessary? I suspect not. However, unless we acknowledge the impact of computers on the goals of mathematics education and try to change the direction of the mathematics program in the secondary years, we will be criticized by future generations for the same reasons cited by an irate teacher over sixty years ago (Sigurdson, 1970):

Come to high school, young people ... True, the material we shall teach you will be completely useless to most of you ... We shall teach it to you anyhow, even if we make incurable bluffers out of you in the process. Our duty in this new day is to train you for citizenship and we propose to perform that duty by administering to you exactly the same training as was given to an entirely different class for an entirely different purpose a generation ago. (p. 414)

Footnotes

- ¹ A mathematics educator told me that she had one education student in her math methods course who was very concerned about the impact on education of "calculators, digital watches, and velcro."
- ² I acknowledge the assistance of Jim Beamer in providing me with a review of the geometry software discussed in this section of the report.

PART 2

REVISING THE SASKATCHEWAN MATHEMATICS PROGRAMS

In this part of the report, changes to the present Saskatchewan mathematics program are proposed in light of the issues raised in Part 1. The program revisions recently proposed in other Canadian provinces - primarily in the West - have been considered. The proposals and recommendations presented in Part 2 are largely in accordance with the recommendations for curriculum reform made by the National Council of Teachers of Mathematics in their recent publication *Curriculum and Evaluation Standards for School Mathematics* (1989). In my opinion, this report will have considerable influence on the development of mathematics programs and associated instructional materials, especially textbooks, in Canada as well as in the United States. The intent of Part 2 is to provide the curriculum writers with a good background in the issues surrounding curriculum development in mathematics and in research related to student understanding of mathematics. Such information should aid the writers as they develop the details of a revised mathematics program for the elementary, middle, and secondary grades.

Chapter 5 provides an overview and critique of the current elementary and middle years mathematics programs. Because the mathematics program for each age group has a very similar organization, the content and issues relevant to the elementary and middle years program will be discussed together rather than separately. Chapters 6 through 14 discuss the following mathematical areas as they apply to a revised elementary and middle years programs:

Chapter 6 - Problem Solving (translation, process, and realistic problems)

Chapter 7 - Foundations of Number Knowledge (part-whole relationships, counting, place value, fractions, decimals, ratio and proportion, and percentage)

Chapter 8 - The Basic Number Combinations

Chapter 9 - Computational Knowledge (paper-and-pencil methods, mental methods, calculator)

Chapter 10 - Estimation (computational and measurement)

Chapter 11 - Measurement (process of measurement, understanding of length, area, volume and capacity, mass, time, temperature, and angles)

Chapter 12 - Geometry and Spatial Sense

Chapter 13 - Data Organization and Analysis (graphing, statistics)

Chapter 14 - Algebra

A discussion of the current secondary mathematics program is presented in Chapter 15 and recommendations for revising the secondary mathematics program are proposed in the final chapter, Chapter 16.

CHAPTER 5

REVISING THE ELEMENTARY AND MIDDLE YEARS PROGRAMS

Before any curriculum and instructional changes are proposed, the content aims and underlying philosophies of the current elementary and middle years programs will be reviewed. In many ways the current elementary and middle years mathematics programs are quite sound. However, some serious deficiencies exist, especially as the curriculum is translated into classroom practice. The reader is reminded that the terms elementary and middle years in Saskatchewan refer to the grade ranges of 1 to 5 and 6 to 9, respectively. The current curriculum documents use the terms Division I, II, and III to describe the grade ranges of 1 to 3, 4 to 6, and 7 to 9, respectively.

An Overview of the Current Elementary Program

The current elementary mathematics program is outlined in the 55-page document *A Curriculum Guide for Division I and II*. Published in 1981, the document contains a list of objectives for each grade from 1 to 6 that an "average student" is expected to master by the end of each school year. These objectives are organized under the headings and sub-headings (or what the curriculum guide terms "strands" and "sub-strands") listed in Table 1.

The content of the program is defined in terms of learning outcomes - behavioral objectives describing measurable student behaviour. For example, the following objectives categorized under the strand Numeration - Whole Numbers indicate the expected student behaviour for year 1 students:

1. associate a set of objects with the corresponding numeral up to 100.
2. identify the one and tens place value positions in any two-digit numeral.
3. compare any two numbers less than 100 as greater than or less than.

Table 1. The Content of the Current Division I and II Mathematics Program

1. NUMERATION	4. GRAPHING
Whole	5. STATISTICS AND PROBABILITY
Fractional and Decimal Numbers	
2. OPERATIONS	6. RATIO AND PERCENT
3. MEASUREMENT	7. PROBLEM SOLVING
Linear	
Area	
Capacity	
Mass	
Temperature	
Time	
Money	

The guide emphasises that the goal of the program is to develop a **focus on understanding** rather than on rote memorization. To meet this goal, teachers are urged to involve students in applying what has been taught, to use concrete and semi-concrete instructional materials, and to provide an instructional environment that promotes active involvement of students.

An important aspect of the program is the **development of favourable attitudes**. Teachers are asked to encourage students to develop an open-mindedness in approaching new problems, a willingness to attack new problems and confidence in applying their computational skills.

Teachers are urged to instruct explicitly students in **thinking processes** associated with transferring mathematical knowledge. The process skills are listed in Table 2.

The definitions of these terms are not given, and the only guidance given to teachers to help students develop these processes is the following statement: "Teachers should construct sub-objectives incorporating the process skills in their daily and long range planning."

Table 2. Process Skills Listed in the 1981 Division I and II Mathematics Curriculum Guide

1. Acquiring information

- 1.1 Observing, listening, direct experiencing
- 1.2 Vicarious experiencing
- 1.3 Locating and using resources

2. Evaluating information

- 2.1 Consistency - logical, empirical
- 2.2 Completeness
- 2.3 Precision

3. Organizing information

- 3.1 Coding, labelling, naming, numbering
- 3.2 Listening, recording, tabulating
- 3.3 Selecting, grouping, classifying, graphing
- 3.4 Sequencing, ordering
- 3.5 Manipulating, estimating, computing
- 3.6 Summarizing, abstracting

4. Interpreting information

- 4.1 Formulating problems
- 4.2 Hypothesizing
- 4.3 Decoding
- 4.4 Inferring, interpolating
- 4.5 Analyzing
- 4.6 Associating, relating
- 4.7 Comparing, contrasting, measuring
- 4.8 Synthesizing
- 4.9 Verifying
- 4.10 Generalizing

5. Communicating information

- 5.1 Expressing results in such a way that others understand
 - 5.2 Using a variety of communication skills
-

The guide discusses **diagnostic-prescriptive teaching**, a model of teaching centering on the diagnosis of errors and the planning of corrective instruction to eliminate these errors. Teachers are told, "When properly

planned, very little teacher time is actually required to administer simple diagnostic tests used to monitor progress or diagnose for particular problems." No examples of 'simple diagnostic tests' are provided to assist teachers in developing suitable instruments.

The guide recommends the following steps be followed when planning diagnosis and remedial activities: (1) identify the specific concepts or skills that a student has not mastered, (2) formulate objectives to serve as a structure for remediation of the diagnosis [sic], (3) plan appropriate learning activities for each objective, (4) monitor and modify the learning experiences in order to meet the needs of the student.

The guide urges teachers to make **problem solving an integral part of the entire mathematics program** and to include daily problem-solving practice activities. Polya's conceptual model for problem solving comprising of the following four stages is briefly discussed: (1) understanding the problem - identifying what is known, unknown, and the problem requirements; (2) planning a promising approach, possibly by recognizing similarities to a previously solved problem; (3) carrying out the plan and, if the plan proves unsuccessful, returning to earlier stages; and (4) checking the solution against the problem requirements. These stages are mainly descriptions of what a good problem solver does. It is questionable, however, whether they can be used to develop a program of instruction, especially since the guide provided teachers with few details.

The guide is more specific in identifying the types of problems that should be the focus of instruction: "In addition to problems found in the textbooks, the teacher is encouraged to develop with students **word problems based on everyday experiences** [bold emphasis is mine] that take into account the students' increasing repertoire of computational skills." Although the guide did not state it explicitly, learning to solve traditional word problems - often referred to as "translation" problems - is to be the focus of problem-solving instruction in the elementary grades.

In an oblique reference to non-routine problem solving, some useful problem-solving strategies are identified: viz., trial and error, patterns, solving a simpler problem, working backward, and simulation. Other than including a list of these strategies, non-routine process problems are not discussed in the guide.

The guide encourages teachers to create a classroom environment "in which problem solving can flourish:" namely an environment in which students question, experiment, estimate, explore, and suggest explanations. The guide does not explain how

to facilitate these student behaviours in a program that emphasizes the solution of straightforward translation problems.

The guide recommends that students be permitted to read problems both silently and orally. No evidence is presented to validate the following claim: "To hear a problem increases understanding and may even give an insight into a probable method of solution."

The guide takes a cautious position with respect to the use of calculators in school. It suggests that the inexpensive portable calculator necessitates a re-examination of standards for computational proficiency and claims that "long and complex calculations will be no longer done manually." However, the computational standards of the program contradict this claim. For example, year 6 students are expected to be able to perform the following types of calculations by hand:

- add and subtract whole numbers up to 12 digits.
- multiply two factors whose product is less than a 12-digit number.
- divide whole numbers containing 'n' digits in the dividend by an 'm' digit divisor ('n' and 'm' representing any number of digits).

By expecting students to perform such calculations, the writers of the document provide teachers with this implicit but unambiguous message: "Teach the same amount and types of computation that have always been taught in the past."

The guide emphasises that electronic aids should not be used to replace student understanding and skill development in mathematical operations and algorithms. It also provides a confusing suggestion that "[calculators] be used to focus upon particular aspects of algorithms and operations such as partial products or partial quotients." The guide does not explain why calculators should be used to calculate only part rather than all of a calculation. The guide also recommends that the calculator be used to check calculations by hand, a pointless procedure. The types of knowledge and skills needed to use a calculator effectively are not identified in the guide.

The guide provides teachers with scant information on the use of computers in mathematics, perhaps because of the lack of availability of microcomputers in schools in 1981. The following rather vague and confused statement is the only one describing the use of computers in the elementary mathematics program: "In the later years of Division II a program [sic] in computer learning environments could be provided to better prepare students to live in a world in which many functions are being done by computer."

The guide provides teachers with these three **textbook options**: (1) using a single textbook, (2) using a number of textbooks, or (3) using no textbook. For the majority

of teachers whose primary instructional source is the textbook, the guide identifies several series recommended by teachers who completed a questionnaire developed by the Department of Education after piloting one of these textbook series.

The guide urges teachers to use these textbooks prudently and emphasises that the prescribed content takes precedence over any scope and sequence defined by the authors of a textbook series. Textbooks are to be considered as teaching aids or resources rather than as a substitute for the curriculum guide.

An Overview of the Current Middle Years Program

Strictly speaking, no middle years mathematics program presently exists. It must be viewed temporarily as a hybrid formed by combining the grade 6 course included in the Division I and II mathematics program and the grades 7, 8, and 9 courses included in the Division III mathematics program. A description of the middle years' program is contained in the 53 page document *A Curriculum Guide for Division III Mathematics* published in September, 1978.

In the *Introduction*, the purposes, philosophy, major objectives and the general features of the program are discussed. The following program goals are listed:

1. To develop 'a positive student attitude towards mathematics based on an understanding and appreciation of its role in society.
2. To develop students' confidence in their ability to do mathematics.
3. To develop a high level of competence in computational skills.
4. To extend the ability to apply basic mathematical concepts, abstractions, and computational skills to problem solving.
5. To extend students' ability to apply basic mathematical concepts and problem analysis skills to the interpretation of their environment.
6. To develop mathematical reading and communication skills.

The guide provides little information on how teachers can help children meet these goals. For example, only one paragraph expands on the last goal: to develop reading and communication skills. Teachers are told that the skills needed to read narrative materials differ from those needed to read mathematical materials and they "must

accept responsibility for teaching the reading skills associated with mathematics." The specific skills needed to read mathematical materials are not identified in the guide.

The *Introduction* provides a list of recommended textbooks for each grade and concludes with a section titled, "The student who experiences difficulty with mathematics." This latter section is too brief to be useful to teachers.

The guide also suggests that the new program will be evaluated: "It should be emphasized that the program is experimental in nature and regular reassessments will be conducted. It is felt that a sound program has been developed and all teachers are urged to adhere as closely as possible to the curriculum guidelines." To the best of my knowledge, no evaluation of the program has ever been undertaken.¹

The second section *Core Topical Outline and Learning Outcomes* identifies the content for each grade level. As in the elementary program, the content is specified behaviourally. The strands and sub-strands used to organize the student learning outcomes are listed in Table 3. As can be seen by examining Table 3, the current program in all middle years is intended to be broad and include topics from many areas of mathematics.

In the third section of the guide, *Correlation of Textbooks with Learning Outcomes*, teachers are provided with a listing of pages in each prescribed textbook whose content corresponds to the learning outcomes for each grade. Pages to be used for enrichment are also identified.

Table 3. The Content of the Current Division III Mathematics Program

1. NUMBER AND NUMBER OPERATIONS	3. GEOMETRY
Notation	
Number	
Operations	
Principles	Polynomial Expressions
Ratio and Proportion	Solution of Open Sentences
	Algebraic Problem Solving
2. MEASUREMENT	5. STATISTICS
Knowledge of SI	
Linear	
Area	
Volume and Capacity	6. PROBLEM SOLVING
Angles	

The current middle years mathematics program for students in years 6 to 9 needs to be changed in several ways. First, portions of the program are very repetitive. After the introductory, conceptual work has been completed in the elementary years, middle years students presently spend much of their classroom time practising computation with larger whole numbers or with numbers different from whole numbers: e.g., computing with decimals and fractions or numbers expressed in scientific notation. Considerable overlap in content exists between the current grade 6 course and the grade 7 and 8 courses.

The overlap in content from one year of the mathematics program to the next as been noted by several writers in other countries as well. Flanders (1987) studied popular American textbooks for grades K - 9 and found that in grades 7 and 8, only about 30% of the textbook pages introduced any new material, the lowest percentage in the K-9 span. (In grade 9 textbooks, the amount of new material rose to 90% largely because the content presentation centred on a new topic, algebra.) Flanders has possibly underestimated the amount of repetition in grades 7 and 8 because many studies (International Study of Mathematics [ISM], B.C. Assessment, NACOME, for example) indicate that textbook chapters containing non-traditional topics such as probability, statistics, geometry are largely ignored by teachers who believe they are "pressed for time." Teachers in Saskatchewan likely ignore many of these topics as well.

The lack of an introduction of new material from year to year can be justified if the intent is to ensure that students develop an integrated view of mathematics. Unfortunately, an examination of the textbooks presently used in grades 6 to 9 indicates that the presentation of identical topics does not change substantially from year to year. The underlying, implicit logic appears to be to introduce difficult topics in the elementary grades and then plan corrective instruction in the junior high grades.

A second weakness of the middle mathematics program is its lack of a consistent framework to select topics for study from year 6 to 9. The current program through years 6 to 8 centres on the learning of arithmetic notions, especially on developing mastery of the standardized pencil-and-paper algorithms. Many of these notions are then ignored thereafter when the study of algebra begins in year 9. This apparent lack of connection of the year 8 and year 9 courses has existed in Saskatchewan since at least 1967.

In the 1967 curriculum guide for Division III (years 7 to 9) mathematics, only the most basic elements of algebra are listed for year 9. Instead of being restricted to algebra, the study of mathematics was broadened to include arithmetic, geometry, measurement, non-decimal numeration systems, sets, and statistics. However, most of the proposed topics were rarely introduced to year 9 students in Saskatchewan because the textbook authorized for the grade 9 and 10 courses presented only algebra. This is another example which illustrates the common observation that textbooks are

usually more influential than curriculum guides in promoting program change or maintaining the status quo.

In Saskatchewan, the present year 9 program continues to be more closely associated with the secondary than with the middle mathematics program and is commonly viewed as a preparatory year for the "serious" study of mathematics included in years 10 to 12 rather than as an extension of ideas presented in earlier years. Secondary teachers in the Saskatoon Public Schools rejected the prescribed course for grade 9 outlined in the provincial document in favour of their own local course. In June, 1986, the document *Grade Nine Algebra Course Outline* prepared by a committee of secondary teachers outlines the core objectives for the grade 9 mathematics course. After a review of "basic skills" (set theory, integer arithmetic, and rational numbers), only algebraic notions are to be introduced: solving equations, polynomials and powers, factoring polynomials, optional topics (graphing on a coordinate plane, geometry constructions, geometry terminology).²

Despite their official inclusion in the Saskatchewan program, many difficult and fundamental ideas involving number and space are usually ignored or neglected, especially by grade 9 teachers. But studies of achievement over the past 50 years have demonstrated that the important mathematical notions involving fractions, rates, ratios, and percentages and their corresponding applications are difficult for many students to grasp without considerable instruction and practice.

Another weakness of the current program is the tendency for middle years' teachers to avoid the use of concrete materials and not to relate mathematics to everyday situations. Middle years' teachers often mistakenly assume that most students are capable of abstract thinking and, therefore, do not need to work with materials. A revised program needs to ensure that teachers know how to use materials effectively to develop understanding in mathematics.

Determining New Program Goals

As an important first step in revising the elementary and middle years mathematics programs, the curriculum writers will have to determine the goals, core content and learning outcomes appropriate for children varying in age from 6 to 15 years. Furthermore, the writers will have to consider how to help teachers cope with any proposed changes that might require a change in their instructional approaches, their understanding of the goals of mathematics education, and their attitudes towards the learning of mathematics.

Based on the literature review on the goals of school mathematics presented in Part 1 and the analysis of the present elementary and middle year mathematics program, I make the following broad recommendations. Each recommendation or set of recom-

mendations will be briefly discussed in this chapter and elaborated further in Chapters 6 through 14.

Recommendation 1

That the elementary and middle years program place less emphasis on learning routine mathematical procedures and more emphasis on solving problems and understanding concepts and that the number of expected learning outcomes be reduced to allow more time for thinking, reading, discussing, and writing about mathematics.

In order to ensure that children have time to develop their understanding of mathematical concepts and processes, a revised program for the elementary and middle years should contain significantly less content than in the present program. Only the most fundamental notions of number and space should be chosen as the core, and other traditional but less fundamental notions should be either eliminated entirely, transferred to the secondary program, or be included as enrichment or optional topics.

In my view, the current elementary and middle years programs contain an excessive amount of content that cannot be mastered by most students, especially in the amount of time provided for the teaching of mathematics. In general, children should not be pressured to master mathematical ideas beyond their capabilities. We need to **plan for success rather than failure** by ensuring that the large majority of students learn some fundamental mathematics well rather than a number of mathematical ideas superficially.

Observations of classroom practice indicate that the lack of instruction aimed at the development of higher order thinking is directly related to a relentless pressure on teachers to attend to an ever increasing list of objectives. Desforges and Cockburn (1987) describe the overcrowded curriculum as a "curriculum kilolitre pouring into a classroom pint pot" (p. 144). Based on the findings of their intensive and long-term research of infant schools in Britain, they concluded that teachers have little choice but to provide a once-over-lightly treatment of basic facts and routines. Consequently, little time remains for a more serious, thoughtful treatment in the teaching and learning of mathematical ideas.

In revising the curriculum for young children, Desforge and Cockburn recommend that "some drastic decisions have to be made about curriculum priorities in the light of judgments about what is reasonable to attain rather than dreams about what is desirable" (p. 144). They recommend the following content form the basis of an elementary school mathematics program:

In the absence of any convincing model of the curriculum as an alternative to that presently practised, we suggest that schools be expected to cover less ground and to use the time saved to cover it more thoroughly. Thorough here means going beyond the reproduction of skilled routines and into the intellectually challenging, practical application work. ... Specifically with respect to mathematics, the concepts of measurement, money and the four rules [adding, subtracting, multiplying and dividing] with small numbers would provide an ample set of issues to engage and develop children's applications skills up to the age of 11. (p. 145)

Although I propose that the amount of content should be reduced and the pace of instruction slowed accordingly, the same mathematical ideas should not be presented repeatedly in a typical drill-and-practice format. Instead, the pace can be slowed in the following productive ways: (1) by ensuring that the basic ideas of mathematics appear in a variety of seemingly different situations or contexts; (2) by encouraging children to use the mathematics they learn to solve problems in a variety of different ways; (3) by providing children with more time to think, discuss, read and write about mathematics. Children and teachers need to appreciate that mathematics involves much more than the production of rapid answers to obvious questions.

In reducing the amount of content to be covered in the elementary and middle years, students **should be exposed to difficult ideas**; the program should not become so barren that children are never challenged, never exposed to failure, and never required to persevere. Abstract ideas should be presented to students but mastery of these ideas should not necessarily be expected. For example, studies show that most year 1 children cannot solve an open sentence presented symbolically such as $4 + ? = 9$. But most can solve a representative problem put in a meaningful context such as the following: "In this box I have placed a coin. Listen to me count as I place another 4 cents in the box. '6, 7, 8, 9.' I now have 9 cents. What coin did I first place in the box?"

Hiebert (1984) maintains that "The critical instructional problem is not one of teaching additional information but rather one of helping children see connections between pieces of information that they already possess." (p. 510). His view should guide the development of the elementary and middle years mathematics program. In a nutshell, a revised elementary and middle grades program should be characterized by the connection rather than the accumulation of mathematical ideas.

Recommendation 2

That the program take into account the developmental level of elementary and middle years students and be based on research findings related to the mathematical reasoning of children and adolescents.

In planning the introductory stages of the school mathematics program, the developmental aspects of learning must be carefully considered. A delicate balance must be maintained between the abstract nature of seemingly straightforward mathematical concepts and the developmental levels of children and adolescents.

On the one hand, children come to school with considerable everyday mathematical experiences: they understand many fundamental concepts, can recall many facts, and are familiar with many mathematical conventions and terminology; and, they are able to perform some routine mathematical procedures such as counting.

Children do not enter school as blank slates upon which school mathematics can be directly inscribed, as was once thought by many psychologists and some educators.³ The knowledge children bring to school can facilitate rather than hinder the mastery of formal mathematics. Baroody (1987) suggests that "the roots of mathematical competence extend back to the preschool years, and successful school instruction builds on this informally learned knowledge" (p. 20).

On the other hand, there are limits to what can and should be expected of students in the elementary and middle years. Carefully planned and well-executed instruction is needed to ensure that students continue to develop powerful mathematical ideas consistent with their levels of cognitive development.

Teachers must be especially careful not to equate mastery of mechanically performed operations on symbols and the recall of memorized factual information with the understanding of mathematical concepts. As Baroody suggests, care must be taken to ensure that children do not master arithmetic at the expense of learning mathematics (p. 44). Often children must arbitrarily memorize ideas they don't understand. This consumes a large amount of instructional time and, as many studies have shown, usually results in a superficial understanding of mathematics.

A teaching demonstration I did several years ago illustrates why memorized mechanical performance should not be equated with true understanding. I asked a small group of grade 3 children to see if they could use the worked out examples below to figure the rule for "diffing" by looking for a pattern and then use the rule to complete the remaining exercises:

$$1. \text{diff } (3y^2) = 6y^1$$

$$4. \text{diff } (4y^2) = ?$$

$$2. \text{diff } (4y^3) = 12y^2$$

$$5. \text{diff } (3y^3) = ?$$

$$3. \text{diff } (2y^3) = 6y^2$$

$$6. \text{diff } (4y^4) = ?$$

A few children worked out the rule correctly by themselves and several more figured out the rule after being given some hints. As this anecdote illustrates, the mechanical

rules for performing some elementary types of differential calculus are well within the grasp of elementary school children, but understanding the techniques of calculus and knowing when to apply them to solve problems are clearly not.

Recommendation 3

That students be given meaningful activities to help them construct their own mathematical knowledge and to allow them to develop their own approaches to solving problems.

Children vary considerably in their knowledge and abilities even before they first enter school, but they all share the common characteristic of "constructing" knowledge: that is, the deeper understandings underlying mathematics are at least partly "invented" by each individual learner (Resnick & Ford, 1981, p. 250). Strictly speaking, number notions cannot be "taught" to students. Grade 1 students must be able to abstract for themselves the characteristics of "fourness," for example. Basing her views on the extensive work into children's learning conducted by Piaget and his followers, Kamii (1985) argues that the typical workbook exercise of having children write the numeral corresponding to a set of pictured objects will not help children learn number concepts because "Number is something children construct by thinking, in their heads, and not by pushing pencils" (p. 5).

Of course, children cannot be expected to construct all forms of mathematical knowledge. They need to be taught explicitly the many conventions of mathematics: writing, reading and saying the various forms of numerals and operations and interpreting mathematical terminology. Children should not be expected to reinvent the conventions and rules developed by humankind over many centuries. Therefore, direct instruction will continue to be an important part of mathematics teaching.

Many mathematical conventions and terminology are often learned best while exploring more interesting mathematical ideas. For example, primary teachers should not teach the terms for various shapes such as 'triangle' by assigning worksheets on shape recognition. Instead, their students should learn the correct term incidentally while engaged in activities exploring the geometric properties of triangles, for example, constructing a mosaic of triangles.

Constructivists do not view children as passive receptacles waiting to be filled with knowledge poured in by the teacher. They see children as active participants in learning. Accordingly, teachers need to provide activities that stimulate the construction of knowledge by children, especially those activities that promote relational thinking and encourage children to develop a personal approach to carrying out tasks and solving problems. Resnick and Ford describe a constructivist's view about learning and teaching in this manner:

It is activity in which errors may be frequent, but these errors are part of the child's attempt to make sense of concepts. Constructive learning involves 'trying out' ideas, testing to see which solutions work and which do not. This requires learning materials and learning environments that provide feedback to the individual on the outcomes of the trials. For Piaget, the kind of feedback that helps in this constructive learning process includes information from both the physical and the social environment. (p. 191)

A sound developmental view of learning considers the maturation of children as a challenge rather than as an insurmountable barrier. Teachers cannot be expected to wait until a child has become cognitively mature before introducing a mathematical notion. According to Resnick and Ford, a more practical approach than waiting for nature to take its course in improving the intellectual development of children is for teachers to pose problems that are slightly beyond the learner's current capabilities but not so far ahead to be incomprehensible. In this manner, the ensuing cognitive conflict, as termed by Piaget, eventually presses children toward new and more powerful ways of thinking. In other words, younger, non-conserving children are best served by exposing them to activities that challenge their presently held views about number, space, and measurement.

Lovell (1971), a Piagetian psychologist and mathematics educator, also questions the wisdom about withholding instruction until children become sufficiently cognitively mature:

It is not in any sense suggested that the child must always be "ready" for a particular idea before the teacher introduces it. The job of a teacher is to use his professional skill and provide learning situations for the child which demand thinking skills just ahead of those which are available to him. It is a question of keeping the carrot just ahead of the donkey's nose. When a child is almost ready for an idea, the learning situation provided by the teacher may well "precipitate" the child's understanding of that idea. (p. 7)

For learning theorists such as Resnick and Ford, and Lovell, children become developmentally ready for mathematical ideas by providing them with many opportunities to challenge their current views of number and space.

Recommendation 4

That the program emphasise the use of manipulatives to develop student understanding at all grade levels. Schools should acquire sufficient

materials to support an activity-based program and teachers should be trained in their most effective use.

In a conceptually-based program, teachers must spend considerably more time than they are presently doing to help children develop concepts and learn to solve problems. Conversely, in the early stages of the mathematics program less time than is spent at present should be spent on the more formalized aspects of mathematics: its symbolic conventions, terminology, and standardized procedures. For many teachers, this change in emphasis will necessitate a significant shift in practice, from using textbooks, workbooks, and exercise sheets that are primarily designed to provide practice in symbol manipulation devoid of any real-world context to using manipulative materials and learning centers.

The environment in which children learn should encourage active exploration and interaction with adults, other children, and materials. The basic learning materials needed to support a sound mathematics program include collectables, prepared materials, and commercially-available materials such as the following:

Collectables: Egg cartons, plastic food containers, boxes, cans, paper rolls, magazines and catalogues, calendars, local maps, counting objects (toothpicks, tongue depressors, pennies, dried beans, pasta, buttons, nuts & bolts), pouring objects (rice, "Vermiculite"), and so on.

Self-prepared: Base 10 beansticks, numeral cards, expression cards, sorting mats, place value mats, overlapping paper plates to represent fractions and angles, crude balances and timers, graphing mats, 100 square, 100 chart, game boards, and so on.

Commercial: Base 10 blocks, plastic counters, attribute blocks, blank playing cards, spinners, wooden and plastic cubes, pattern blocks, dice, coloured plastic figures for counting, sorting and patterning (animals, cars, etc), geoboards, play money, clocks, rulers, graduated containers, weighing balance, 3-d geometric models, interlocking cubes, scales, thermometers, rulers, tapes and other types of linear measurement devices, transparent plastic grids, various types of abaci (hoop, bead), calculators, reading materials of varying length and complexity, games, suitably-designed interactive computer software, and so forth.

Concrete materials are viewed by many educators as an intermediate representation between real-world objects or events and symbolic representations of them (Hiebert, 1984). The empirical support for the importance of concrete materials is sketchy because past research has focused more on whether or not the materials improve

performance: for example, whether work on a decimal abacus improves performance on the addition algorithm.

Children also reach the stage where a physical analog is not necessary and may even be a hindrance to learning. Anyone who has tried to explain how to use Cuisenaire rods to divide fractions is familiar with the limitations of concrete materials. Once mathematical ideas become sufficiently familiar and 'concrete' to children, they can be used to help children learn new and more advanced mathematical ideas. For example, if a child understands that multiplication and division are inverse operations, this understanding can be used to help in the understanding of the division algorithm for whole, decimal and fractional numbers.

Clearly, those teachers who use only a textbook and workbook will need to be provided with sufficient motivation to change their current instructional practices and will have to be supplied with some manipulatives and certainly a variety of instructional ideas. Both preservice and inservice education will have to help teachers learn how to use manipulatives effectively.

Often teachers are unclear about the purposes of manipulatives and become so concerned about children working with manipulatives they forget why they are being used in the first place. Manipulatives are used for the following purposes:

1. To help students develop a mental image or model of a mathematical idea.
2. To help students develop appropriate language describing mathematical ideas.
3. To help children practice previously learned skills in a meaningful context.
4. To help children perceive mathematical relationships.
5. To help provide children with an interesting and challenging learning environment.
6. To help children learn through as many senses as possible.
7. To provide teachers with a way to evaluate the deeper understandings that are often not measured by pencil-and-paper work alone.
8. To teach mathematical ideas that are difficult if not impossible to teach through explanation alone.

Teachers also need to become familiar with the following suggestions about the use of manipulatives in the mathematics program:

1. Manipulatives are used until the children become familiar with the underlying mathematical ideas and the associated language.
2. The same type of manipulative can be used with different ages of students to teach different mathematical concepts by modifying the nature of the instructional activities and tasks.
3. Although for reasons of interest, it is best for children themselves to handle the materials, manipulation is not always necessary in learning a mathematical idea.
4. Instruction must proceed in a systematic manner from concrete, to pictorial, to symbolic work.
5. Step-like procedures such as algorithms are best learned by connecting each symbolic step with a corresponding physical action rather than proceeding with all symbolic steps and then trying to relate these steps to all the physical actions.
6. Grouping allows children to communicate and exchange ideas while reducing the amount of material that needs to be made available.
7. Not all manipulatives equally represent the intended mathematical idea.
8. Whenever possible and appropriate, students should be solving problems or engaged in a game-like situation when using manipulatives.
9. If one manipulative is used to teach a mathematical idea, another should be used to review the same idea.

Recommendation 5

That the mathematics program be modified to ensure that students develop positive beliefs and attitudes towards the learning of mathematics.

Recent research suggests that children quickly form beliefs about school mathematics that are quite resistant to change and can influence negatively what is learned in later courses in mathematics (see Carpenter, Hiebert, & Moser (1983); Cobb (1985); Lester (1983)). Baroody (1987) argues that schools implicitly or, in some cases, even explicitly

convey a message to children that it is important to learn and use the correct facts and to do so quickly. This message fosters the following unrealistic, perfectionist beliefs about learning mathematics (pp. 68-69):

- An inability to learn facts or procedures quickly is a sign of inferior intelligence and character.
- An inability to answer quickly or use a procedure efficiently indicates "slowness."
- An inability to answer correctly denotes a mental deficiency.
- An inability to answer at all signals real stupidity.
- All problems must have a correct answer.
- There is only one (correct) way to solve a problem.
- Inexact answers (e.g., estimates) or informal, unconventional procedures (e.g., trail-and-error problem solving) are undesirable.

Unreasonable beliefs about mathematics discourage sound yet unconventional thinking. They encourage slavish devotion to mechanical procedures even when such procedures are inefficient and unneeded.

The 1985 B.C. Mathematics Assessment (Robitaille & O'Shea) found that students believe mathematics is important and relatively easy to learn but they do not enjoy their experiences in learning mathematics. The majority of grade 7 students, for example, disliked learning about estimation, the metric system, working with perimeter and area and solving word problems, topics believed to be important by mathematics educators. A revised program should aim at improving student attitudes about the learning of mathematics. This aim will be realized only if teachers are provided with instructional ideas that will appeal to them and their students.

Recommendation 6

That the mathematics program be integrated whenever possible with other areas of school learning without sacrificing its quality.

Ideally, the learning of children should not be compartmentalized into narrowly defined subject areas. Children should learn some mathematics while they are engaged in activities that might be labelled as "science," "social studies," or "visual art." This view of learning is consistent with the notion of the Common Essential Learnings (Saskatchewan Education, 1988). Learning in the traditional content areas

should promote numeracy, the ability of students to function effectively in quantitative situations.

Students should learn and practice mathematics outside as well as inside the "mathematics period." Similarly, in the mathematics period, students should be provided with planned experiences in all four language modes: listening, speaking, reading and writing. The mathematics period should not exclude language experiences because "One learns to write as one learns to speak as one learns to read and listen. Each ability improves to the extent that all are exercised." (Saskatchewan Education, 1988, p. 12).

Recommendation 7

That the program not be designed around a single authorized textbook and that teachers be provided with sufficient materials to support a resource-based program.

The curriculum writers need to be aware of the powerful influence on curriculum and classroom practice exerted by textbooks. Many studies indicate that textbooks are more influential than curriculum guides in defining a program and, without sufficient resources, preparation time, and experience in program development, few teachers go beyond the prescribed text. In other words, the textbook, rather than the curriculum guide, eventually directs the content, goals and instructional methods of the program.

NACOME (1975), for example, found that 56% of American K-12 math teachers used a single textbook, while only seven percent of the respondents believed they were sufficiently confident to depend on their own resources and use no basic text. NACOME also found that the teachers followed closely the lessons in the text and rarely did they expect the students to read the textbook. One wonders whether textbooks are written for teachers or for students.

The study conducted in British Columbia as part of the *Second International Mathematics Study* found that textbooks and worksheets produced by teachers are among the most significant determinants of the curriculum as it is actually taught (Robitaille, O'Shea, & Dirks, 1982). Although the study was confined to students and teachers in grades 8 and 12, the findings likely apply to all other grades. The authors found that many teachers indicated that they would be reluctant to follow approaches other than the ones presented in the textbook but they claimed that they did not select teaching approaches based primarily on the approaches present in the text. The authors concluded, "if certain topics, approaches, or orientations are to be implemented in the mathematics curriculum, care must be taken to ensure that these are included either in the prescribed textbooks or in other instructional materials which are made available to teachers" (p. 162).

The NCTM *Standards* (1989) made the following statement about textbook design and the influence of textbooks on the implemented curriculum:

We are aware that the curriculum in many schools is geared to textbooks, and we expect the standards to be used as criteria for measuring text content. However, we do not believe the standards can be met simply by altering current texts (e.g., appending a genuine problem at the end of each chapter or inserting a chapter on statistics). Those are quality-control solutions to a problem that demands a design solution. Nor do we believe that textbooks should drive instruction. Rather, other materials that support the standards, such as manipulatives and courseware, must be developed in addition to textbooks. (p. 252)

Saskatchewan Education must provide leadership and encourage textbook publishers to develop new textbooks that accurately reflect the goals and intent of a revised elementary mathematics program.

Recommendation 8

That calculators be available in the elementary and middle grades and that students be instructed in their proper use.

Saskatchewan Education must strongly support this recommendation, which evolved from the discussion in Part 1, if the mathematics curriculum is to reflect the technological realities of today rather than those of the past. We have to ensure that teachers realize that the current expectations for computational proficiency in the elementary and middle years are unrealistic, especially in light of the inexpensive and abundant portable calculator. As the B.C. Royal Commission states, "The use of calculators, except for simple computations, should become routine and all students should be provided with calculators as they are now provided with textbooks" (p. 98).

Recommendation 9

That students in the elementary and middle years be exposed to elements of the following topics: problem solving, foundations of number, computation including calculators, written and mental calculation and the recall of the basic number combinations, estimation, measurement, geometry and spatial sense, data organization and analysis, and algebra.

Each topic listed in Recommendation 9 is discussed in great detail in Chapters 6 through 14. It is not necessary to organize the curriculum under all the topics, however. Topics such as problem solving, data organization and analysis, estimation,

measurement, and likely geometry and spatial sense, should be integrated into the program rather than taught as separate units. Perhaps the entire elementary and middle years curriculum can be condensed into only three topics, number, space, and algebra. Regardless of how the curriculum is organized, we need to ensure that it is taught in an integrated fashion and that children see the relationships between these mathematical topics and know how to apply mathematics to real-life situations.

Footnotes

- ¹ I was a member of the Division III Mathematics Curriculum Committee from its inception in 1977 to its dissolution in 1988.
- ² According to the mathematics consultant of the Saskatoon Public School Board, a committee struck to review the grade 9 mathematics course is preparing a revision that more closely coincides with the current provincial course.
- ³ Thorndike (1922) considered children so deficient in mathematical ability that he concluded: "It seems probable that little is gained by using any of the child's time for arithmetic before grade 2, though there are many arithmetic facts that can be learned in grade 1" (p. 198).

CHAPTER 6

PROBLEM SOLVING

A common refrain in any recent publication in mathematics education is that problem solving should be the focus of the elementary school mathematics program. The following statements are typical:

Learning to solve problems is the principal reason for studying mathematics. (National Council of Teachers of Mathematics, 1980)

Problem solving has been integrated within all strands and materials have been selected to support the teaching of problem solving. Two criterion for the selection of textbooks will be the quality and variety of problems presented. (B.C. Ministry of Education, 1988)

Problem solving must be an integral part of the entire mathematics program and as such should be practiced on a daily basis. (Saskatchewan Education, 1981)

These views about the centrality of problem solving in the mathematics curriculum are certainly not new. An examination of earlier curriculum guides, journals and reports indicate that problem solving has always been a high priority.

Obviously, the improvement of problem solving requires more than chest-thumping rhetoric. As teachers continually find, it is not easy to help children become good problem solvers because problem solving is a very personal kind of intellectual activity that is influenced by a wide variety of interrelated factors. Lester (1983) draws an analogy between learning to ride a bike and learning to solve problems. First, it is difficult to explain either activity in words. In describing bike riding to a beginner, one might say, "Get on the bike, push off with one foot and then pedal like mad while balancing yourself." A beginner would not find this description helpful in any way. Second, watching another person ride a bike usually does not aid the beginning bike rider. A third parallel between bike riding and problem solving not raised by Lester is that practising the component skills of mounting, balancing, pedalling, and turning in preparation for the act of riding is no guarantee that the learner will be able to coordinate these skills when riding.

Problem solving in mathematics is similar to bike riding insofar as no description can express its complexity, subtlety and highly personal nature. The ability to solve problems is often not fostered by imitating others who solve problems and, in some

instances, modelling can be counterproductive. (We all know students who complain that they know how to do problems until they have to do some on their own.) Because teachers typically are never seen to struggle with a solution, show only correct moves, and seemingly grab ideas out of mid-air, many students believe that problem solving is a mysterious process beyond their capabilities. Finally, practising the component skills (computing, recall of basic facts, writing equations and number sentences) beforehand does not ensure that these skills will be applied when a problem is encountered.

What is a mathematics problem? Lester (1980) suggests that for a situation to be a problem for an individual, several criteria must be satisfied (p. 30): (1) the person must be **aware** of the situation; (2) the person must be **interested** in resolving the situation; (3) the person must be **unable to proceed directly** to a solution; (4) the person must make a **deliberate attempt** to find a solution. The important point here is that a situation considered a "problem" for one person is not necessarily a "problem" for another. Consequently, it is difficult to state unequivocally that a given statement or situation will or will not be a problem without considering the nature of the problem solver.

What types of problems should be included in the mathematics curriculum? A revised mathematics curriculum for all ages and abilities of students should contain a good mix of the following types of problems: (1) translation problems, (2) process problems, and (3) realistic problems.

Translation Problems

Ask most teachers to give an example of a mathematics problem and they will likely quote a series of statements taken from their mathematics textbook. The following problem taken from *MathQuest 5* is typical: "An orchestra gave a series of 8 summer concerts. They sold an average of 2000 tickets for each concert. How many tickets were sold in all?" These types of problems, hereafter referred to as translation problems,¹ are intended to elicit a particular automatic response when presented to students: In the case of the problem cited from the text, teachers would expect the student to carry out in some manner the multiplication of the numbers 8 and 2000.

Translation problems possess several distinguishing features. First, they are self contained; that is, they contain precisely the information necessary to solve the problem. The problem solver does not need to seek out any further information. Second, translation problems usually have only one acceptable answer: "about 280 dollars" is frequently not considered an acceptable answer according to the standards set by textbook writers, teachers, test developers and even most students. Precision is a tacit, yet important, requirement when translation problems are presented in school.

Third, there is an expectation that translation problems must be solved by using standardized procedures, usually some previously taught rule or algorithm. It is expected that students will approach all translation problems in the following manner: correctly interpret the problem, choose some procedure from among a collection of known procedures (usually the most recently taught one), and carry out the procedure using the given numerical data (Borasi, 1986). For example, following a lesson on working with proportions, teachers in the middle grades would expect the students to set up an equation and solve it in the manner presented below:

$$\begin{array}{l} \$340 = x \\ 40 \text{ h } 1 \text{ h} \end{array} \quad \begin{array}{l} \text{--- } 340 \times 1 = 40 x \\ \text{--- } 340/40 = x \\ \text{--- } x = 8.5 \end{array}$$

In most cases, before proceeding to assign another similar "problem," the teacher would not accept nor explore alternative methods of solution. Consequently, some students believe that there are 'right' ways and 'wrong' ways to solve problems. Usually the right way can be identified by examining the most recent lesson presented in the textbook or done by the teacher on the chalkboard.

Translation problems along with "mathematical exercises" (mathematical expressions such as $55 \times 45 = ?$) form the large majority of "problems" that children are presently required to solve in school. Translation problems are considered surrogates for real-world problem solving, preparing children to solve problems encountered in daily life. There is, however, little evidence to support this view (see Kilpatrick, 1985). Failure to solve translation problems does not necessarily imply that students will not be able to solve problems or carry out routine mathematical tasks seen outside of school. Many studies have demonstrated that there is little relationship between solving translation problems and solving everyday problems.

Translation problems can, at best, provide children with opportunities to interpret commonplace expressions as mathematical statements and as such they might help in developing an understanding of mathematical concepts. When children correctly respond with statements like, "I added 4 and 3 and got 7; 7 cents is my answer," or write down the statement $4 + 3 = 7$ after being presented a translation problem that incorporates the operation of addition, we have some confidence that they understand at least one aspect of adding. Whether or not they are able to carry this understanding into problems seen in daily life is another matter.

It can be argued that translation problems are not really problems at all. If one characteristic of a problem is that the problem solver is unable initially to proceed directly to a solution, a translation problem cannot be considered a problem when its solution is immediately obvious. In the early stages of development, translation problems fit the requirements of problems for most children. In the later stages of development, when children begin to respond mechanically, translation problems are best thought of as algorithms rather than as problems. Despite their limitations,

translation problems are still an important part of school mathematics because they require students to recognize situations where an operation might be applied.

But students of all ages need to be provided with more than translation problems if we expect them to become good problem solvers. When children are limited to the solution of one-step translation problems commonly seen in current programs, they often respond mechanically and their performance on other less routine problems is severely curtailed. For example, Lester (1985) reports that grade 3 and 5 children experience considerable difficulty solving non-routine or process problems such as the following:

Tom and Sue visited a farm and noticed that there were chickens and pigs. Tom said, "There are 18 animals." Sue said, "Yes, and they have 52 legs in all." How many of each kind of animal were there? (p. 41)

Almost all grade 3 children added 18 and 52, while most of the grade 5 children divided 52 by 18. When questioned about their methods of solution, a typical response of the third-graders was something similar to "It asks 'how many in all,' so you add." Fifth-graders typically said, "It says 'in all' but you don't add because that would be legs plus animals. The key words are 'how many of each,' so you divide."

Lester reported that even when the students were told that their answers were incorrect, they were unable to devise an alternative procedure for attacking the problem. Few children tried to understand the problem by drawing a simple diagram, enumerating a few cases, or making a table. The typical reaction was simply to search for some key words that seemed to suggest the correct operation, usually those appearing in the last sentence or problem question. Perhaps this behaviour is a consequence of the narrow range of problems presented in school. Although many teachers continue to teach the key word approach to children, research has demonstrated that this approach has little to offer because key words can mislead as much as they illuminate.

The problem-solving approaches of children can be greatly influenced by surface characteristics irrelevant to the solution of a problem. For example, a 12-year-old girl in Melfort, who was given a multi-step translation problem to solve, immediately responded, "I can't do that problem because there aren't two numbers. I can't get any questions without two numbers (Hope & Baker, 1986, p. 7). Despite encouragement from the interviewer and the use of probing questions, the child refused to tackle the problem any further.

When presented with a standard translation problem, many children spend the majority of the time computing solutions rather than analyzing problems. A 12-year-old was presented with the following problem:

Mrs. Jones travelled 12 625 km in her car during the first year she drove it. During the second year, she drove 9 862 km. How many kilometres had she travelled in those two years?

She arranged and subtracted the two numbers on a sheet of paper. She chose subtraction rather than the correct operation of addition "Since Mrs. Jones had some mileage that had a bigger number and some that had a smaller number." Most of her solution time was spent in copying the numbers down neatly and performing the various steps of the algorithm instead of trying to make sense of the problem.

Another 11-year-old girl provided this explanation of her problem-solving approach: "Problem solving is easy. If there are more than two numbers, I always add; otherwise, I subtract. If I'm not sure if it's multiplication or division, I divide and, if there's a remainder, I multiply instead" (Hope & Baker, p. 7). This type of student likely exhibits frustratingly erratic problem-solving performance because her approach would yield correct answers for some problems presented in school and incorrect answers for others.

Kieren and Olson (1989) provide a good example of how the ability to translate words into mathematical statements is not necessarily a good measure of a student's understanding. A grade 7 girl hesitantly wrote the expression $y + 3y = 24$ when presented with the following problem:

Tom drank several glasses of milk. Al drank three times as many glasses of milk. Together they drank twenty-four glasses. How many glasses did each drink? Can someone write an expression for this situation? (p. 14)

When the teacher asked the girl to solve for y , she wrote the expression $4 + 3(5) = 24$. To help her identify her error, the teacher (thankfully a novice) responded by suggesting that the distributive property be applied, confusing the student and her classmates even more.

Because problem solving in school is virtually limited to the solution of one-step translation problems, children view all problems as "words disguising an operation." Children rarely take the time to read and understand a problem before they plunge into a calculation. They avoid considering alternative methods of solution and only occasionally draw diagrams, make a table, substitute straightforward numbers to simplify the problem statement or employ other useful strategies to gain insight into the problem. Why should children bother with any thoughtful consideration and reflection if they are provided with problems requiring only addition, subtraction, multiplication and division? When dealing with problems presented in school, children quickly learn that the most efficient approach is to choose an algorithm,

usually the one that has been most recently taught and practised, and then search for some numbers to be calculated. If the wrong numbers or operations are chosen initially, contextual cues rather than mathematical understanding can often be used to modify a problem-solving approach.

This tendency of children to unmask the operation imbedded in a problem and then use the context of the problem to select another approach was observed in a 12-year-old girl when she was presented with the following problem (Hope & Baker, p. 9):

85 girls were arranged in lines in the gym with 5 girls in each line. How many lines were there?

Her four attempts to solve the problem and her accompanying comments are presented below:

Computation

Child's comments

$$\begin{array}{r} 85 \\ - 5 \\ \hline 80 \end{array}$$

85 - 5 = 80. Hmm. That doesn't sound right.
Let's try multiplication.

$$\begin{array}{r} 85 \\ \times 5 \\ \hline 425 \end{array}$$

425 rows in the gym? That can't be right!

$$\begin{array}{r} 85 \\ + 5 \\ \hline 90 \end{array}$$

Hmm. 90? That still sounds like too many lines in a gym. Yuck. I guess I'll try division.

$$\begin{array}{r} 17 \overline{) 85} \\ 5 \overline{) 85} \\ \underline{5} \\ 35 \\ \underline{35} \\ 0 \end{array}$$

17 remainder 0. That sounds better. But I still don't think it's right. Oh well, who cares.

This child never attempted to represent the problem such as drawing a rectangular arrangement containing several lines of 5 objects. Neither did she attempt to reduce the complexity of the problem by substituting smaller whole numbers for the quantities expressed in the problem. She did try to verify the reasonableness of the solution, however, by using her knowledge of the size of the school gym. Without denigrating her sound technique for checking the reasonableness of a calculated solution - at least, she attempted to see if her answers were reasonable - her initial answers would have

been considered satisfactory if the context of the problem involved a larger area than her gym, a field house, for example.

Even successful students experience difficulties in knowing when to apply a computation to some types of translation problems, especially when the problem involves fractions. For example, the National Assessment found that only about 20% of 13-year-olds could solve a problem like the following:

Mary lives $\frac{2}{5}$ of a mile from school. How far has she walked (in miles) when she walks $\frac{2}{3}$ of the way to school?

This same study also found that only 21% of 17-year-olds could solve the same problem; not much of an improvement for an additional four years of schooling.

To see if this finding applied to Saskatchewan students, I have administered this item over the past several years to several hundred second-year elementary education students attending the University of Saskatchewan. These students have all completed Mathematics 102.6, an introductory university mathematics course, as well as the high school prerequisite Algebra 30 and most have also completed Geometry-Trigonometry 30. Typically, although almost all can add, subtract, multiply and divide fractions (about 5% have forgotten how to divide fractions), only about half recognize that this problem can be solved by multiplying $\frac{2}{3}$ and $\frac{2}{5}$. Of the students who cannot solve the problem, about 30% choose division and the remaining 20% choose subtraction.²

These students are neither mathematical novices nor academically weak (because of competitive enrolment quotas their high school grades are higher than many students studying in other Colleges). Yet they continue to have difficulty knowing when to apply a computational technique. One student expressed her frustration to me by remarking, "I passed Math 102 but I'm not sure whether I should multiply or divide. How can I ever explain to children how to do such problems?"

What can be done to improve the ability of students of all ages to solve translation problems? As Lochhead and Mestre (1988) caution, no quick fix exists: "Although many mathematics teachers are familiar with the general translation difficulty, most are perplexed by its persistence. We have studied it intensely for nearly a decade but remain astonished at how difficult it is to teach students to avoid the common pitfalls" (p. 131).

To improve the teaching of translation problems, the following recommendations are made:

1. Promote understanding of concepts.
2. Provide opportunities for students to create and solve their own problems.
3. Encourage the careful reading of problems.
4. Vary the semantic structure of problems.

Concept Development

Currently, the typical approach used by teachers and textbook writers to develop the ability to solve translation problems is to ensure that students can perform the mechanical procedures supposedly needed to solve problems and then hope that through continual exposure to translation problems the problems will be solved in the intended manner. Consequently, instruction in most classrooms is aimed at developing and practising the so-called "fundamentals" (namely, computing and recalling basic number facts) rather than developing concepts and problem-solving skills.

When mathematical concepts and operations are introduced they should often follow rather than precede problem-solving opportunities. For example, the formal aspects of division should be introduced to elementary school children only after they have had sufficient opportunities to analyze problems like the following:

Grandmother visits and gives four brothers and sisters 13 dollars to share equally. How much money does each child get?

Before the formal terms and symbols are presented, students should learn to approach this type of problem in a variety of ways: for example, by role playing, by simulating using counters, or by drawing a representative diagram. There should also be considerable discussion about how other methods such as repeated addition, repeated subtraction or even multiplication can be used to solve the problem. When the approaches to this problem have been exhausted, the teacher (and the text) should vary the problem slightly by changing the numerical values - the money to be shared and the number of children - and then encourage the children again to use a variety of problem-solving approaches. Similar problems that represent a 'sharing' process should then be presented for analysis. Finally, when children can solve these types of problems with confidence, the division symbol and the mathematical terms 'divided by' and 'division' can be introduced.

Teachers should also understand that the connection between an algorithm or procedure and a translation problem is not readily apparent to many children. The following instructional progression is recommended to help students see these connections. Consider the NAEP problem discussed earlier:

Mary lives $\frac{2}{5}$ of a mile from school. How far has she walked (in miles) when she walks $\frac{2}{3}$ of the way to school?

Students do not see the relationship between this problem and the computation $\frac{2}{3}$ multiplied by $\frac{2}{5}$.³

Teachers have to recognize that this problem requires an understanding that one number represents a distance ($\frac{2}{5}$) while the other number represents an operator ($\frac{2}{3}$). Problems such as the following can help students make this distinction:

Mary lives 6 miles from school. How far has she walked (in miles) when she walks $\frac{1}{3}$ of the way to school? Explain how a number line can be used to solve this problem.

Similar problems should be considered by gradually changing the distance to other whole number units and by keeping the operator $\frac{1}{3}$ constant. Students should discuss how these distances can be partitioned into 3 equal parts, perhaps thorough folding paper strips representing the distance. The problems should then progress to using whole number distances but the operator should be changed to $\frac{2}{3}$.

By examining the solutions, students should become aware that the operator $\frac{2}{3}$ is equivalent to doubling the operator $\frac{1}{3}$. The final set of problems should involve the unit fraction distance $\frac{1}{5}$ of a mile and progress to the intended distance $\frac{2}{5}$ of a mile. In each case, the operator should remain constant at $\frac{2}{3}$. Such a gradual but controlled progression increases the likelihood that students perceive the relationship between this problem and the multiplication of two fractions.

Teachers should also draw attention to the relative size of the numbers in the problem and the ensuing answer: that is, a fractional operator less than 1 reduces the size of the operand and, consequently, the answer must be less than $\frac{2}{5}$ of a mile. Such thinking helps students determine the reasonableness of the answer and teachers should constantly probe for such understanding.

The students can then be presented with other related problems representing a different conceptual view:

Mary had $\frac{3}{4}$ of a pizza remaining. She sprinkled hot sauce on $\frac{2}{3}$ of the remaining pizza. What fraction of the original pizza contained hot sauce?

Student-Created Problems

Student-created problems help students associate mathematical procedures and concepts with translation problems. Young children can be asked to base their created problems on pictures and real world objects, which eventually can be passed on to other children to solve. For example, a teacher can ask students to create a mathematical problem based on the situation where each of three children are given a nickel. Older children can be asked to create problems based on given number sentences such as $9 + 5 = 14$ or $1\frac{1}{2} \times \frac{2}{3} = ?$

Generally, children find it difficult to construct a meaningful problem based on a computational situation. O'Brien and Casey (1983) found that about a third of the sixth grade students participating in their study could not make up a story problem given the expression $6 \times 3 = 18$, although multiplication is introduced in grade 3. The two stories written by a grade six boy in Saskatoon provides a good illustration of this difficulty. Asked to write a story to represent the number sentence $625 \times 25 = 15\,625$, he produced the following:

We were driving to a far city and it took 625 miles to get there and it took 25 miles to get through the city. How many miles did we travel altogether?

His second story represents the number sentence $3 \times 12 = 36$:

Dad was building a playhouse and it was 3 metres wide and 12 metres long. How many metres would be the area?

The additive problem he created to represent the first number sentence is a typically incorrect response. Yet when smaller numbers were used in the second number sentence he successfully created a problem to correspond to one type of multiplication situation.

Reading of Problems

Students rarely find it necessary to read carefully the problems presented in school. They quickly acquire an attitude that reading is a bothersome matter because most problems are solved by applying the most recently learned mathematical procedure. This attitude is often carried into high school.

A German study found that children in grades 2 through 6 do not read problems carefully. Instead they immediately do some figuring when presented with the statements like the following:

Mr. Lorenz and 3 friends started out at Bliefeldt at 9 A.M. and drove the 360 kilometres to Frankfurt with a rest stop of 30 minutes.

Younger children, who did not enjoy the benefits of problem-solving instruction in school, patiently waited for the interviewer to provide further information.

To ensure that children read problems and give them thoughtful consideration, the wording, information given, and questions asked should frequently vary. The following variations are recommended:

1. Vary the problem question. The problem statement should be varied to call for more than a calculated answer. For example, the routine problem "John has 10 dollars and Freda has 13 dollars. How much money do they have altogether?"

"is less thought-provoking than the modified problem "John has 10 dollars and Freda has 13 dollars. Do they have enough to buy mother a 25 dollar gift certificate? If not, how much more money do they need?"

2. Provide incomplete problems. Problems with insufficient or superfluous information such as those below require careful reading to determine the relevant information:

1. Bill pays for candy with a dime. How much change will he receive?
2. There are 16 crayons of 8 different colours in a box. Jim takes 6 crayons away. How many crayons remain in the box?

Studies have shown that children find it very difficult to solve these types of problems.

3. Use mixed sets of problems. Except in the introductory stages of learning, sets of practice problems that all employ the same operation should be avoided as they require little thinking - merely operating on numbers found in the problems. Instead, mixed problem sets like the following should be used:

1. Solve only those problems where the fact $10 + 10 = 20$ can be used.

Ten people have 10 dollars each. How much money do they have altogether?

Ten people share 10 dollars equally. How much money does each person get?

Bill's mom and dad each gave him a dime. How much money did they give him?

2. Identify those problems suggesting the equation $3x + 2y = 50$.

The total number of wheels in a shop selling power trikes and motorcycles equals 50. How many trikes and motorcycles are on display?

There are 50 vertices in an art design made up of triangular and rectangular shapes. How many of each kind of shape are used in the design?

In sorting their 50 tape cassette collection, Millicent sorted hers into 3 boxes while Alphonso sorted his into 2 boxes. How many tapes do they each have?

4. Vary the wording. Students need to be provided with problems in which the wording is varied. Teachers know that the wording of a problem can have a significant effect on its difficulty. Obviously, if children do not understand the wording, they will be unable to solve the problem.

Although it is well known that a difference in wording contributes to a problem's difficulty, it is not exactly clear how. Carpenter (1985) claims that other than knowing that certain wordings are more difficult than others, we do not have a clear pattern of how differences in wording influence children's solutions (p. 26). Obviously, the difficulty of a problem like the one below is not necessarily reduced simply by ensuring that children understand all the terminology incorporated in the problem statement (pewter, tin, lead, kilograms, parts) or that the problem passes some test for readability typically applied to narrative material.

Pewter is 70 parts tin and 30 parts lead. If 200 kilograms of tin is used for an order of pewter, how much lead must be used?

Moreover, it is difficult to know whether the following parallel problem containing nonsense words would be easier or more difficult to solve than the original problem:

Mooga is 70 parts tinsy and 30 parts plubonium. If 200 poonds of tinsy is used for an order of mooga, how much plubonium must be used?

To help children separate the wording from the underlying mathematics, they must be exposed to many similar problems with slightly different wordings. A study reported by Carpenter shows how word problems can be greatly simplified if the language is made more transparent. Pre-school and grade 1 children were shown pictures of groups of birds and worms and asked two questions requiring a comparison of quantities:

1. How many more birds than worms are there?

2. Suppose the birds all race over and each one tries to get a worm. Will every bird get a worm? How many won't get a worm?

Most children were unable to solve the first problem but could easily solve the related second problem, despite its greater wordiness and syntactic complexity.

Semantic Structure

Students need to work with problems that vary in semantic structure. Most one-step translation problems seen in the elementary and middle grades are characterized by teachers and textbook writers as involving either addition, subtraction, multiplication, or division. That is, children are expected to recognize one of these operations imbedded in a given translation problem. Consequently, teachers and textbook writers typically present children with sets of problems classified by operation. Research has demonstrated that this may be an inadequate way to categorize problems. The two problems below, for example, can both be classified as 'subtraction' problems, but children often use two different strategies to solve them neither of which may involve 'subtraction.'

1. John had one dollar. He bought a candy bar for 58 cents. How much money did he have left?
2. John has one dollar. Mary has 58 cents. How much more money does John have than Mary?

The first problem can be represented by removing a set of 58 objects from a set of 100 objects - a type of subtractive process usually referred to by teachers and children as "take-away." (In real life, it would likely be solved not by subtracting, but by adding or counting on from 58 to 100.) The second problem can be represented by comparing, through a 1-1 correspondence, a set of 100 objects with a set of 58 objects - another "subtractive" process. Clearly, it is an oversimplification to classify both problems as subtraction and research demonstrates that children find it more difficult to see the second problem as subtraction than they do the first problem.

Recently, researchers (Carpenter, & Moser, 1984; Carpenter, 1985; Thompson & Hendrickson, 1986; Hendrickson, 1986) have been categorizing translation problems according to their semantic structure rather than on their underlying mathematical operation to determine the relationship between these problem structures and students' problem-solving approaches. Thompson and Hendrickson (1986) identified three broad classes of problems that involve additive and subtractive processes: 1) Change, 2) Combine, and 3) Compare. Tables 1, 2 and 3 illustrate how simple one-step problems have been classified according to these structures.

Change Problems

All types of Change problems involve two distinct actions: namely increasing or decreasing an initial set to create a final set. Change problems have three quantities expressed, one of which is unknown: an initial set, a change set and a final set. Thus, as can be seen in Table 1, six kinds of Change problems can be created by varying the unknown quantity.

Combine Problems

Combine or Part-Part-Whole problems involve static relationships between a larger group and two or more subgroups. Combine problems can be distinguished from Change problems in that there is no direct or implied action on the groups and subgroups. Combine problems can be separated into two kinds. In one kind, the quantities of the two subgroups are known; in the other, the quantities of both one subgroup and the larger group are known.

Compare Problems

Compare problems involve the comparison of two distinct groups: the referent group and the compared group. The amount that exceeds the quantity of the smaller group is termed the difference. Table 3 illustrates six types of Compare problems. The unknown quantity can be the referent set, the compared set, or the difference set. For each of these three possibilities, the comparison can be stated in two ways: (1) the (larger) compared set is **more than** the (smaller) referent set, or (2) the (smaller) compared set is **less than** or **fewer than** the (larger) referent set (Thompson & Hendrickson, p. 23). Compare problems are probably the most ignored of all types of problems in school mathematics.

Hendrickson has attempted to categorize the various problems that can be represented by multiplicative and divisive actions. Tables 4 through 7 includes the classification scheme and some sample problems.

Table 1. Additive and Subtractive Change Problems (Taken from Thompson & Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Change 1	Bill has two pencils. Jean gives him three pencils. How many pencils does Bill have then?	Increase, initial set and change set known, question about final set

Change 2	Bill has five pencils. He gives three to Jean. How many pencils does he have left?	Decrease, initial set and change set known, question about final set
Change 3	Bill has two pencils. Jean gives him some more. Now he has five. How many did Jean give him?	Increase, initial set and final set known, question about change set
Change 4	Bill has five pencils. He gives some to Jean. Now he has two. How many did he give to Jean?	Decrease, initial set and final set known, question about change set
Change 5	Bill has some pencils. Jean gave him two more. Now he has five. How many did he begin with?	Increase, change set and final set known, question about initial set
Change 6	Bill has some pencils. He gave three to Jean. Now he has two. How many did he begin with?	Decrease, change set and final set known, question about initial set

Table 2. Additive and Subtractive Combine Problems. (Taken from Thompson & Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Combine 1	Bill has three red pencils and two green pencils. How many pencils does Bill have all together?	Two subsets are known, question about whole set
Combine 2	Bill has five pencils. Three are red and the rest are green. How many are green?subset	Whole set and one subset are known, question about other

Table 3. Additive and Subtractive Compare Problems. (Taken from Thompson and Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Compare 1	Bill has two pencils. Jean has five. How many more does Jean have than Bill?	Comparison stated in terms of "more," referent set and compared set known, question about difference set
Compare 2	Bill has two pencils. Jean has five. How many fewer pencils does Bill have than Jean?	Comparison stated in terms of less (fewer), referent set and compared set known, question about difference set
Compare 3	Bill has two pencils. Jean has three more than Bill. How many pencils does Jean have?	Comparison stated in terms of more, referent set and difference set known, question about compared set
Compare 4	Jean has five pencils. Bill has three fewer pencils than Jean. How many pencils does Bill have?	Comparison stated in terms of less (fewer), referent set and difference set known, question about compared set
Compare 5	Jean has five pencils. She has three more pencils than Bill. How many pencils does Bill have?	Comparison stated in terms of more, compared set and difference set known, question about referent set
Compare 6	Jean has two pencils. She has three fewer pencils than Bill. How many pencils does Bill have?	Comparison stated in terms of less (fewer), compared set and difference set known, question about referent set

Table 4. Multiplicative and Divisive Compare Problems (Taken from Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Compare 1	Joellen has 3 pairs of sandals. She has 4 times as many pairs of shoes. How many pairs of shoes does she have?	Referent set and many-to-one correspondence known; question about the comparison set.
Compare 2	Irene has 30 pennies. She has 5 times as many pennies as Pat has. How many pennies does Pat have?	Comparison set and many-to-one correspondence known; question about the referent set.
Compare 3	Donald has 5 marbles. Peter has 15 marbles. Peter has how many times as many marbles as Donald?	Comparison set and referent set given; question about kind of (many-to-one) correspondence.
Compare 4	Bonnie has 16 white blouses and 4 colored blouses. Her colored blouses are what (fractional) part of her white blouses?	Comparison set and referent set given; question about kind of (one-to-many) correspondence.
Compare 5	Our class has 16 boys and 12 girls. There are how many times as many boys as girls?	Comparison set and referent set given; question about the (many-to-many) correspondence.
Compare 6	Our class has 16 boys and 12 girls. The girls are what (fractional) part of the boys?	Comparison set and referent set given; question about many-to-many correspondence.
Compare 7	Fred has 25 baseball cards. He has $\frac{5}{4}$ as many cards as Jim has. How many baseball cards does Jim have?	Comparison set and many-to-many correspondence given; question about referent set.
Compare 8	Erica has 25 stickers. Peggy has $\frac{4}{5}$ as many stickers as Erica. How many stickers does Peggy have?	Referent set and many-to-many correspondence given; question about comparison set.

Table 5. Multiplicative and Divisive Selection Problems (Taken from Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Selection 1	Paula has 3 kinds of cheese and 2 kinds of sausage. How many different cheese-and-sausage pizzas can she make?	Number given from which to select for each pair element; question about number of pairs possible.
Selection 2	Frank makes 18 different cheese-and-sausage pizzas. He has 6 kinds of cheese. How many kinds of sausage does he need?	Number in one choice set and number of pairs given; question about number in other choice set.
Selection 3 extended n-tuple	Dave has 3 different-sized sets of wheels, 4 kinds of bodies, and 3 different motors. How many different cars with wheels, a body, and a motor can he put together?	Number given from which to choose for each portion in n-tuple; question about number of n-tuples possible.
Selection 4 extended n-tuple	Dave has 3 different-sized sets of wheels and 4 kinds of bodies: he can make 96 different cars with wheels, bodies, and motors. How many different kinds of motors does he have?	Number given from which to choose for all but one position in n-tuple and also number of n-tuples; question about remaining position.

Table 6. Multiplicative and Divisive Rate Problems (Taken from Hendrickson, 1986).

Problem title	Sample problem	Characteristics
Rate 1	Lisa buys 18 cans of polish at \$0.72 per can. What is the total cost? dependent variable.	Given the rate and the independent variable value; question is about the dependent variable.
Rate 2	Peter buys a suit on sale. The price, after a 25% discount, is \$90. What was the original price? variable.	Given the rate and the dependent variable value; question is about the independent variable.
Rate 3	Corrine runs 200 metres in 72 seconds. What is her average speed in metres per second?	Given the values of the dependent and independent variable; question is about the rate.

Table 7. Multiplicative and Division Change Problems (Taken from Hendrickson, 1986).

Problem title	Sample problem	Characteristics
(Change 1) Repeated addition	Fred has 3 boxes with 4 cars in each box. How many cars does Fred have? set.	Initial (smaller) set sizes and change number known; question about final (larger) set.
(Change 2) Repeated subtraction (measurement)	Jean had 12 cookies. She gave 3 cookies to each of her friends. How many of	Initial (larger) set and final (smaller) set sizes known; question about change number. her friends got cookies?
(Change 3) Partitioning into equal sets	Paul had 24 marbles that he gave away to 4 friends. Each friend received the same number of marbles. How many marbles did each friend get?	Initial (larger) set and change numbers known; question about the size of final (smaller) sets.

Observations of children solving these various types of problems indicate that (1) children use different processes to solve problems differing in structure and (2) some types of problems are more difficult to solve than others. In general, the inherent structure of the problem seems to be the crucial factor in determining its difficulty (Thompson & Hendrickson, p. 23). For example, a Change problem such as number 1 in Table 1 is structurally straightforward and easily solved by young children. But a Change problem such as number 3 in Table 1 where the quantity of one subset is unknown is conceptually more complex and, consequently, more difficult. Children need to have a well-developed sense of part-whole relationships to tackle the latter type of problem.

Children initially solve these various addition and subtraction problems by directly modelling the action or relationships described in the problem. For example, in solving a problem such as "Connie has 5 marbles. How many more marbles does she need to have 13 marbles altogether?", most children will use initially some type of additive strategy (adding on from 5 until 13) rather than use the subtractive strategy

($13 - 5 = 8$) considered by textbook writers and teachers to be the 'proper' way to solve this type of problem.

Carpenter and Moser (1984) found that for many children the structure of the problem influences the strategy at least through the third grade, despite the considerable attempts of primary teachers to ensure that children can recall addition and subtraction number facts. It may be several years before children learn to solve problems simply by recalling a number fact or applying some algorithm. Moreover, Carpenter (1985) argues that the requirement of using a school-learned procedure often hinders the development of the ability to solve problems:

Even before they have received formal instruction in arithmetic, almost all children exhibit reasonably sophisticated and appropriate problem-solving skills in solving simple word problems. They attend to the content of the problem; they model the problem; they invent more efficient procedures for computing the answer. Given the limits of their mathematical skills, this performance is remarkable. Contrast this with performance several years later, when many children solve any problem by choosing a single arithmetic operation based on the surface details of the problem. (p. 37)

Although these schemes for categorizing the majority of translation problems seen in elementary school are primarily used as a research tool, there are at least two ways that they can help in planning the elementary and middle years curricula.⁴

First, they can be used to create a better balance of problems than is currently seen in elementary mathematics classrooms. Stigler, Fuson, Ham and Kim (1986) assessed the impact that the problems contained in textbooks might have on the relative difficulty of addition and subtraction word problems. They compared the word problems presented in four American series with those in a Soviet series at the first-, second- and third-grade levels. All the series were widely used and published in the 1980s.

To classify problems presented in these series, they used the Carpenter and Moser scheme, which is similar to the Thompson and Hendrickson discussed earlier. The findings of Stigler et al. have clear implications for textbook writers and teachers. First, the Soviet series included many more two-step problems than the American series: 44% of the problems in the Soviet series versus only 7% in the American series. There is a richer range of two-step combinations presented in the Soviet texts, and many such problems (half the total) are given in the first year of school (p. 161). Second, the distribution of problems across the grades varies considerably from series to series. The Soviet texts contain a relatively even mix of problems according to semantic type. The American texts have a very uneven distribution of problem types. Sample comparisons are indicated below:

1. Compare problems are rarely evident in the American series while the Soviet texts contain a roughly equal number of the six types of problems.
2. The Soviet texts present all six types of Change problems, with fairly equal distribution across four of the six types. In contrast, the American texts present a very large number of only two of the six change problems and either none or a very small number of the remaining four types.
3. Examination of the four most numerous problem types shows that the American texts emphasise the easier problem types; viz., those ones in which the arithmetic solution procedure directly parallels the semantic structure of the problem.

The researchers hypothesise that the relation between problem difficulty and semantic type prevalent in the literature may be partially explained by their frequency of presentation. As textbook makers probably are motivated to give teachers material that will not be too difficult to teach, they over-represent (probably unwittingly) the easiest problem types. Furthermore, the researchers argue that causality might be working in both directions, "leading to a snowball effect in which children are asked to solve easier and easier problems, which in turn makes the difficult problems even more difficult and thus less likely to appear in the elementary mathematics textbooks" (p. 169). The researchers suggest that a more sophisticated and less haphazard approach to problem selection be undertaken by textbook writers, curriculum developers and teachers.

Classifying problems by semantic type rather than by operation also suggests ways to help children solve problems. By knowing which problems are the most difficult, teachers can spend their time more profitably by concentrating their efforts on these problems rather than on the problems that children can already solve. Teachers can use this knowledge to build on what children know rather than forcing them to adapt solution strategies not understood. For example, children should be taught explicitly that there are a number of ways to solve Compare problems such as the one below:

Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?

Children should be given opportunities to approach this problem in a number of ways including the following:

1. Compare a set of 13 objects with a set of 5 objects by counting onward from 5 to 13.

2. Compare the sets by adding on in larger steps such as adding to 10 and then counting on 3 more or similarly adding to a sum they know and then counting on.
3. Compare the sets by counting backward or subtracting in steps to determine "how many fewer than" rather than "how many more than."
4. Compare the sets by adding on to 5 (i.e., 5 and 8 more equals 13) or by subtracting ($13 - 5 = 8$).
5. Recalling [eventually] a basic fact ($13 - 5$) to figure the difference.

Students find it especially difficult to recognize the underlying structure of translation problems that involve rational numbers (Bell, Swan & Taylor (1981); Fischbein, Deri, Nello & Marino, (1985)). Bell et al. have shown that when children are presented with the same content presented in a series of translation problems, their choice of operation is influenced by the specific numerical data contained in the problem. For example, 12- to 15-year-olds in Britain were asked to indicate the operation that they would use to find the cost of 0.22 gallons of petrol if one gallon cost 1.20 pounds. (They were told not to worry about a calculation.) The most common answer was "divide 1.20 by 0.22." However, when the same question was asked with whole numbers, such as 2 pounds for the price of a gallon and 5 gallons for the amount of petrol, the pupils answered correctly, $2 \times 5 = 10$. When interviewed, the students did not believe that it was inconsistent to choose a different operation when numbers are changed. Their reasoning was something like "0.22 gallons of petrol must be smaller than the cost of one gallon, and since division makes numbers smaller, division is the appropriate operation."

Fischbein et al. extended the work of Bell et al. and studied grade 5, 7 and 9 Italian students' responses to translation problems involving the multiplication and division of decimals. Their study investigated the following hypothesis:

... arithmetical operations are intuitively associated with some primitive behavioural models whose existence and influence the person may not be aware of. Such implicit models, acting to a great extent beyond any conscious formal control, may sometimes facilitate the course of problem solving, but very often they may slow down, divert, or even block the solution process when contradictions emerge between the model and the solution algorithm. (p. 14).

The following items used in their study and a brief discussion of their findings may help clarify the meaning of their hypothesis:

- A. With 75 pinks you can make 5 equal bouquets. How many pinks will be in each bouquet?
- B. The walls of a bathroom are 280 cm high. How many rows of tile are needed to cover the walls if each row is 20 cm wide?
- C. To wrap 5 equal packages requires 3.25 m of string. How much string is needed for each package.
- D. The walls of a bathroom are 3 m high. How many rows of tile are needed to cover the walls if the width of each row is 0.15 m?

All four problems can be solved using division: viz., dividing 75 by 5 in A, dividing 280 by 20 in B, dividing 3.25 by 5 in C and dividing 3 by 0.15 in D. Yet the percentage of students in every grade correctly choosing division differed significantly in each problem: for example, 99% of grade 9 students chose division in problem A, 80% in B, 84% in C, and only 55% in D. The most frequent error in D was to multiply 0.15 by 3.

The researchers concluded that students were influenced by, what they termed, "primitive" mental models for division and multiplication. They interpreted division as either being a **partitive** situation (a sharing process where a collection of objects is partitioned into a number of equal sets) or a **quotitive** situation (repeatedly subtracting quantities to determine how many groups of a given size are contained in a collection). These models are usually introduced formally to students in the elementary grades.⁵ Students find it difficult to see how these models apply to non-whole numbers, however. For example, both problems B and D theoretically involve quotitive division. Most students are comfortable with the idea that, when the divisor is a whole number, division means dividing into an equal number of sets: e.g., dividing 20 into 280 to find 14 rows of tile as in problem B. But, it is another matter, to convince them to divide a quantity by measures that are not whole number units: e.g., dividing 3 by 0.15 to find 20 rows of 0.15 m tiles as in problem D.

Fischbein et al. caution that teachers of arithmetic face a fundamental dilemma:

One the one hand, if one continues to introduce the operations of multiplication and division through the models described above, one will create - as our findings demonstrate - strong, resistant, and, at the same time, incomplete models that soon will come to conflict with the formal concepts of multiplication and division. On the other hand, if one tries to avoid building the ideas related to arithmetical operations on a foundation that is behaviourally and intuitively meaningful, one certainly will violate the most elementary principles of psychology and didactics. (p. 15)

The authors report they are presently studying instructional methods to help students overcome the primitive models that seem to control their behaviour.

Their study demonstrates that it is an inadvisable practice for teachers to continue to emphasise calculation over the analysis of problems. Teachers need to spend more time helping students to consider a variety of ways to solve translation problems rather than trying to impose procedures whose relevance is never clearly perceived by students. It is likely much more profitable for teachers to help students diagram or role play a variety of possible ways to solve difficult problems like D (the bathroom tiling problem discussed previously) rather than practising the computation supposedly needed to solve D: for example, determining the number of strips of paper of length 0.5 m, 0.25 m, and 0.15 m needed to cover lengths of 1 m, 2 m, and 3 m. Teachers can then help students develop their own approaches to solving similar problems based on the informal reasoning developed in these preparatory activities: e.g., reasoning that $10 \times 0.15 = 1.5$, and doubling this result [$20 \times 0.15 = 3.0$], and concluding that 20 rows of tile are needed to cover a wall 3 m high.

In summary, translation problems have to be selected more carefully than they have been in the past. Clearly, it is not sufficient to select problems based merely on the mathematical operation that a well-versed adult might choose in solving the problems. I also emphasize that simply listing "learning outcomes" in a curriculum guide will do little to improve problem solving. What can a teacher do with statements such as the following taken from the present elementary curriculum guide?

The student should be able to:

1. verbally analyze a simple word problem.
2. verbally state the operation required when given a simple word problem.
3. solve problems containing more than one step.
4. solve problems containing several operations.
5. solve problems involving volume or circumference.

The problem-solving practices of children will change only if teachers are provided with sufficient training and suitable materials.

Also, teachers need to spend more time helping students make the connection between informal methods they understand and the standard methods that have been formalized over the centuries. Time typically spent on practising routines such as figuring calculations and solving equations presented in workbooks, worksheets and textbooks is better spent on analyzing how translation problems can be solved in a number of different ways. By continually relating formal symbolism to meaningful situations, there is a greater chance that children will make sense of the formal mathematics we eventually expect them to acquire.

Process Problems

Process problems, as the term suggests, are non-traditional mathematical problems where the emphasis is on the solution process instead of on the solution. Process problems cannot normally be solved using routine procedures; rather, their solution typically involves the application of some problem-solving strategy or heuristic. Process problems act as vehicles to encourage the development of strategies that students can apply to other process problems, and all mathematical problems in general.

Process problems are not intended to be necessarily 'practical'; instead, they are used as opportunities for students to devise creative methods of solution, to share their methods with others, and to build confidence in mathematical problem solving. Lester (1985) also believes that process problems may help reduce impulsiveness on the part of children, especially the tendency to "crunch" numbers and generally fail to gain a good understanding of the requirements and conditions of the problem before beginning to solve the problem.

The discussion below focuses on one classic process problem along with some strategies that might be used to solve it.

There were 10 people at a party. If each person shook hands with everyone else only once, how many handshakes were there in all?

The stated problem can be tackled by students in the elementary, middle, and even secondary grades by simply varying the instructional approach. With a class of elementary school children, for example, the problem could be introduced by bringing 10 children to the front of the class to simulate or role play the handshaking procedure suggested in the problem. A pattern becomes evident when the first child is seen to shake hands with 9 people, the second child with 8 people, the third child with 7 people, and so on.

The problem could be further analyzed by completing a table of information gathered from the role play of differing numbers of children and the children could be challenged to find a pattern and to incorporate additional terms into the table:

Number of	1	2	3	4	5	6	7	8	9	10
people										

Number of			
handshakes	0	1	3

To make the problem more manageable, the number of children could be reduced, to say 4, and the handshakes could be organized into a list as shown below:

<u>Bill</u>	<u>Henry</u>	<u>Mary</u>	<u>Joan</u>
Henry	Mary	Joan	
Mary	Joan		
Joan			

This problem illustrates that process problems differ from translation problems in a number of important ways. First, a solution is not readily apparent even to a mathematically sophisticated individual, unless of course the problem is familiar to the problem solver. Consequently, a process problem requires much more thinking and much more time for analysis than the typical translation problem. Second, although the exact solution may require some computation (a knowledge of addition, in this case), knowing how to compute does not guarantee that a solution to this problem will be found. Third, the exact answer to a process problem is not especially important. Once a solution method is discovered by a problem solver, the final numerical answer is often disregarded. In the case of the handshake problem, once the additive pattern is discovered, there seems to be no compelling reason to calculate a final answer. This feature contrasts with translation problems where the focus is usually on producing an exact numerical answer. Fourth, a number of strategies can often be employed to facilitate the discovery of a solution method, although the use of a problem-solving strategy does not necessarily guarantee a solution. But unsuccessful attempts frequently provide new insight into the characteristics of the problem. Fifth, some strategies seem more suited to some problems than others. For example, diagramming, role playing, and making a list seem to suit the handshake problem, but the guess and check strategy somehow seems unsuited. The ability to sense which strategies seem to "fit" a problem and which don't fit is an important aspect in learning to think mathematically and it develops only through a great deal of experience in solving non-routine problems.

The research literature suggests that elementary and middle grades students can improve their ability to solve process problems when the following conditions exist: (1) Problem solving steps are emphasised; (2) a good mixture of problems are regularly presented.

Steps in Solving Problems

Curriculum developers have recently begun to incorporate the various stages that good problem solvers go through when solving mathematical problems. The components (skills and understandings) of each stage are then used as guidelines to select appropriate instructional experiences for children. In this chapter, three stages are identified: (1) **Understanding** - the stage where the problem solver attempts to

understand and get a "feel" for the problem; (2) **Planning** - the stage where a plan of attack is formulated and carried out; (3) **Reflecting** - the stage where a solution has been obtained and checked for its adequacy and where generalizations are made or new problems are considered.

It should be noted that most problem solvers do not proceed necessarily in this order; they can, for example, plan an initial approach, only to find that they didn't quite understand the problem and have to retrace their steps to increase their understanding. By being aware of these stages, teachers and curriculum developers can plan appropriate problem-solving activities for children.

Stage 1. Understanding. Stage 1 encourages students to understand the problem, its requirements, its constraints. In this stage, students are expected to demonstrate this understanding in a variety of ways:

- a. by representing the problem in some manner such as drawing an appropriate diagram or restating the problem in their own words.
- b. by clarifying the problem through asking key questions, identifying key words, recognizing missing or extraneous information, recognizing subproblems.
- c. by determining whether a problem has a solution.
- d. by determining the reasonableness of given information, a given estimate, and a given solution.

Middle years students should work occasionally with vaguely formed, open-ended problems such as the following:

In a village there are 3 streets. All the streets are straight. Each crossroad has one lamppost. How many lampposts are needed? How many are needed for a village with 20 streets? Generalize to any number of streets.
(Varga, 1988, p. 100)

The late Tamas Varga, a Hungarian mathematics educator known for his work in curriculum reform, strongly advocated that students be given problems that can be interpreted in different ways. However, as illustrated by the following anecdote, teachers will have to be given guidance in presenting these types of problems.

A mathematician interviewed by a national newspaper complained of the following ambiguous problem given to his young nephew (Varga, 1988, p. 295):

Somebody tells a joke on Monday to five persons. Next day, Tuesday, each of the five tells the joke to six other persons. Each of the latter tells it to seven persons on Wednesday. How many will have heard it on Wednesday?

The boy found three solutions, depending on the problem's interpretation:

1. Five persons heard the joke on Monday, five times six or 30 on Tuesday, thirty times seven or 210 on Wednesday. The answer is 210.
2. In another interpretation the answer is $5 + 30 + 210$ or 245. Those who heard the joke on Monday or on Tuesday will have heard it on Wednesday together with the 210 who heard it precisely that day.
3. He who told the joke to the first five persons must have heard it previously - unless he invented it - so the answer is **at least** 246.

Instead of being pleased with the quality of his own thinking, the boy expressed the following concern: "The teacher may have in mind another solution, and she will make a fool of me before the class because I could not find the **real** solution. The whole class will laugh at me!" (p. 295). The mathematician concluded (wrongly, I argue) that "Problems in math workbooks should be more carefully worded so as to exclude different interpretations."

Varga maintained that the ambiguous problem was **not** the cause of the student's difficulty but rather his authoritarian teacher was. He agreed with the mathematician that problems should be unambiguously worded "so long as there are authoritarian teachers." If teachers are expected to use and help children profit from open-ended problems, obviously they need to be well informed about the goals of a revised mathematics curriculum.

Stage 2. Planning. In this stage, the problem solver decides on an approach to the problem. The selection of an approach is often based entirely on the problem solver's past experience with solving seemingly similar problems. Helping children recognize similarities between problems is an important goal of mathematics education. Most researchers believe that students need to have explicit instruction in the problem-solving strategies listed below. These strategies should be considered as important learning outcomes in a revised elementary and middle years program.

1. Represent the problem situation with an appropriate diagram, model or simulation.
2. Incorporate the data into an organized list or table or construct a suitable graph.

3. Guess and check: substitute a guessed number, carry out a calculation, check the suitability of the answer, make an adjustment and repeat the process.
4. Reconstruct and solve a simpler problem.
5. Look for a numerical or geometrical pattern that can be generalized.
6. Use logical thinking to eliminate possibilities.
7. Work backwards.
8. Change your point of view.

To clarify the meaning of most strategies, I will present some examples of suitable problems. It should be emphasised that problems are rarely solved through the use of a single strategy. Typically, they are solved when strategies are used in concert with each other. Thus, most of the sample problems presented here can be used to represent the use of many different strategies.

1. Representing or Modelling. Good problem solvers usually construct some representation of the problem in an attempt to understand its main features. This skill needs particular attention because most children will not take the time to draw even a simple diagram or sketch. Younger children have to learn that representations do not necessarily have to be images: i.e., to represent a problem involving dogs, it is not necessary to worry about the anatomical detail of dogs, and symbols can be used instead.

Minnie made triangular shapes using seashells she found at the beach. She laid down 1 shell, followed by a row of 2 shells, followed by a row of 3 shells, and so on. How many shells would she need to make a triangle of 20 rows?

2. Organizing. Most problems containing a large amount of data, especially those involving a potentially infinite process, can be better understood when the data is organized in some manner, usually by producing an organized list, table or graph. Most students need explicit instruction and practice if this strategy is to be mastered.

In how many different ways can three people divide 25 pieces of candy so that each person gets at least 1 piece?

3. Guessing and checking. The guess and check strategy is perhaps one of the most powerful and easily understood problem-solving strategies. I have found that older students who have forgotten how to use some previously taught algebraic method

and younger students who have never been taught any formal method can learn to solve problems suited to the guess-and-check strategy.

Teachers, especially secondary mathematics teachers, often discourage students from using the strategy, despite its intuitive appeal, because they fear it will encourage "guessing." They also tend to think that this strategy is "bad mathematics." However, students who use the guess-and-check strategy often gain greater insight into the nature of a problem than when they use a more formal method they don't understand. Teachers who do encourage the use of the guess-and-check strategy need to ensure that their students know how to use it efficiently.

A cube has a volume of 15 625 cubic centimetres. Using multiplication on a calculator, figure out its dimensions.

4. Simpler problem. When students recognize that a simpler but related problem might be used to solve a more complex problem, they exhibit a great deal of understanding. The strategy can be used in two ways. First, in quantitative problems, smaller or more easily handled numbers are substituted for other numbers in the problem. A surprising number of students and adults do not use this apparently obvious strategy. They seem to use whatever numbers are in the problem and, after struggling with some calculations, usually give up. Moreover, as discussed earlier (see Fischbein et al.), problems involving decimals and fractions are not necessarily simplified by substituting whole numbers.

Second, sometimes the solver recognizes an analogous or related problem that is easier to solve or that has been solved previously by the solver. The solution of the related problem is then applied in a parallel fashion to solve the original problem. Often good problem solvers when they notice that "it's one of those problems" ignore the problem thereafter because the challenge is gone. This strategy is very sophisticated and in some ways might be "unteachable" because it depends on the prior knowledge of the solver and their ability to recognize common properties of problems. Perhaps it is only learned through practice in problem solving and by comparing one's solution attempts with other problem solvers.

The sample problem below illustrates the use of this strategy:

A $100 \times 100 \times 100$ cube is constructed from individual unit cubes. All faces of the larger cube are painted black. If the cube is then dismantled into unit cubes, how many will have 0 faces, 1 face, 2 faces, and 3 faces painted black? Do any have more than 3 faces painted?

5. Patterns. Problems can often be solved by looking for patterns. This strategy is often used in conjunction with other strategies. Because the recognition of patterns is so

content-dependent, it is difficult to teach students this strategy. Students likely acquire the ability to use this strategy through regular practice with problems in which patterns are evident, through discussion with other problem solvers, and through the continual acquisition of mathematical knowledge.

There are 16 football teams in the NFL. To conduct their annual draft, a direct telephone line is connected to each city's team. How many lines must be installed to accomplish this?

6. Logical reasoning. Generally speaking, logical reasoning is involved in the solution of any process problem. It is most evident when the problem solver tries to eliminate seemingly possible solutions as in the sample problem below:

Each letter in the following cryptarithm represents a digit from 0 to 9. Find the value of each digit.

$$\begin{array}{r} \text{SEND} \\ + \text{MORE} \\ \hline \text{MONEY} \end{array}$$

Logical reasoning is used to identify, confirm and reject possibilities: for example, the value of M cannot exceed 1, nor can it equal 0; thus, the digit must equal 1.

An interesting (and difficult) problem that many people are convinced has no unique solution is the following:

During a recent census, a man told the census-taker that he had three children. When asked their ages, he couched his reply with a riddle: "The product of their ages is 72. The sum of their ages is the same as my house number." The census-taker, after taking down the house number and working some calculations, complained, "I still can't tell." The man replied, "I forgot to tell you, the eldest one likes chocolate pudding." The census-taker gleefully wrote down their ages. How old are the children?

7. Working backwards. Working backwards is not a generally useful strategy because it can be applied to so few problems. If it is considered similar to inverse deduction, then it has many more uses. That is, the solver asks questions like: "What do I need to know to show this statement is true." The following problem can be used with elementary school children:

Sam the squirrel found some acorns. He stored half of his acorns in a hollow tree trunk. He then gave half of what he had left to his mate. After that he had 16 acorns. How many acorns did Samuel find?

8. Point of view. Some problems seem difficult because we inadvertently impose conditions that are not contained in the problem statement. In this case, we must learn to change the point of view. The following problems require some flexibility of thinking:

1. Jeff and Barbara are both good chess players. They have completed 5 games, and each has won an equal number of games. There were no ties. How might this have been done?
2. Two bicycle riders are 50 km apart, approaching each other at a speed of 30 and 20 km per hour, respectively. A fly starts from one rider, flies to the other rider, and continually flies back and forth between riders at a constant rate of 80 km/h until the two riders meet each other. How far did the fly travel?

Studies have shown (see, for example, Charles & Lester (1984)) that, given the opportunity, even young children can learn some of these strategies. Informal comments from elementary school teachers in Saskatoon who participated in a problem-solving project I helped develop indicate that children who have been regularly presented with process problems seem to change their problem-solving behaviour: specifically, they seem more willing to try a general approach (e.g., draw a diagram, make a list) when encountering an unfamiliar problem and more likely to persevere. The teachers also reported that the students began to talk about the strategies: for example, "I used guess and check first, and then I drew a diagram."

Stage 3. Reflecting. Good problem solvers are not content simply to obtain a solution. Instead, they usually reflect on the nature of the problem and their method of solution. In this stage, we should encourage the following behaviours:

- a. checking the reasonableness of the solution: numerical estimation, the logic, the correct units, etc.
- b. explaining the solution to someone else.
- c. identifying another possible approach.
- d. making the solution more generalizable.
- e. recognizing similarities to other problems.
- f. creating a related problem by varying the conditions of the original problem.

This final step is frequently ignored by teachers. In a revised program, students should be given the time and encouragement to reflect on their successful and unsuccessful approaches to solving problems.

Presenting a Mixture of Process Problems

A stumbling block in the teaching of problem solving in the past has been a lack of good problems available to the classroom teacher. Until recently most textbooks included few problems, especially non-routine process problems. This omission is unfortunate because most teachers rely almost entirely on a single textbook for their instructional ideas. If teachers are expected to focus on problem solving and to change their current instructional approaches, they must have access to a variety of good problems. Thus, a teacher needs to have this reasonable question answered: Where do I find a selection of good problems?

Fortunately, the recent interest in problem solving has spurred the publication of a variety of books on problem solving, most of which contain a collection of good problems. Saskatchewan Education should develop a list of published sources of process problems appropriate for elementary and middle years students.

Since most teachers, especially those teaching in the elementary and middle grades, do not consider themselves authorities on problem solving, it is quite natural for them to look to other more knowledgeable people for a source of good problems. Typically, they depend on the expertise of the authors of textbooks and other types of resource material. But teachers need to be persuaded that they, as well as many of their students, have the ability to create their own good problems by varying the conditions of other good non-routine problems and of typical textbook exercises.

Charles and Lester (1982) recommend that new, related problems can be formed easily by following a set of five principles (p. 51). The "handshake" problem can be used to generate several other "new" problems:

At a party I attended recently, I noticed that every person shook hands with each other person exactly one time. There were 12 people at the party. How many handshakes were there?

After this problem is presented and discussed, teachers typically move to another completely different problem and quickly exhaust their source of problems. However, with a little effort, several variations can be developed:

1. By changing the context.

Twelve students in Ms. Palmer's fifth-grade class decided to have a table tennis tournament. They decided that each student would play one game against each other student. How many games were played?

2. By changing the **size and type of numbers**.

At a party I attended recently, I noticed that every person shook hands with each other person exactly one time. There were 20 people at the party. How many handshakes were there? What if there were no people at the party?

3. Change the number of conditions.

At a party I attended recently, I noticed that every person but the host, Tim, shook hands with each other person exactly one time. Tim shook hands with everyone twice (once when they arrived, once when they left). There were 12 people at the party. How many handshakes were there?

4. Reverse **given and wanted** information.

At a party I attended recently, I noticed that every person shook hands with each other person exactly one time. If I told you that there were 66 handshakes, could you tell me how many people were at the party?

5. Change some **combination** of context, numbers, conditions, and given/wanted information.

All 20 students in Ms. Palmer's fifth-grade class decided to have a table tennis tournament. They decided that each student would play one game against each other student. How many games were played?

As can be seen, it is relatively easy to generate new and interesting problems based on a good initial problem. Perhaps, with sufficient experience, students can be taught and encouraged to develop variations as well.

Most textbooks present students with relatively mundane problems to solve, but the skillful teacher can create some interesting variations by changing some of the original problem requirements. For example, consider the following variation created from a standard problem taken from recently published mathematics textbooks:

1. Textbook problem/situation: An array of Canadian stamps is illustrated to represent multiplication combinations.

The variation: The pictures show blocks of 1 cent, 2 cent, 6 cent, 8 cent, 10 cent, and 25 cent stamps. What are the least number of stamps needed to send the following letters: 37 cents? 43 cents? 74 cents? \$1.56?

Realistic Problems

One of the major goals of mathematics education is to ensure that students acquire the ability to apply the mathematics they learn in school to solve real-life problems. As the following extract presented in the Cockcroft report (1983) contends, the content of mathematics courses does not always seem to relate to students' experience of everyday life:

Mathematics lessons in secondary schools are very often not about anything. You collect like terms, or learn the laws of indices, with no perception of why anyone needs to do such things. There is excessive preoccupation with a sequence of skills and quite an inadequate opportunity to see the skills emerging from the solution of problems. As a consequence of this approach, school mathematics contains very little incidental information. A French lesson might well contain incidental information about France - and so on across the curriculum; but in mathematics the incidental information which one might expect (current exchange and interest rates; general knowledge on climate, communications and geography; the rules and scoring systems of games; social statistics) is rarely there, because most teachers in no way see this as part of their responsibility when teaching mathematics. (p. 142)

The Cockcroft report suggested that, although there is strong pressure for mathematics courses to be more "relevant to the requirements of everyday life," this aim is easier to state than achieve. In referring to a study of secondary school practice in Britain, the Cockcroft report comments that "it was very common for the schemes of work [curriculum guides] in the schools visited to refer to the need to relate the mathematics taught to the problems of everyday life, but the convincing realism of this aim was much more rare" (p. 141).

Why is it seemingly difficult for teachers to convince students that mathematics is relevant and meaningful? There are several reasons that many students, with the exception of those who enjoy mathematics for its own sake, are not convinced that mathematics "is about something." First, as has been discussed earlier in this report, the problems facing the majority of students when they leave school require little knowledge of mathematics. Most students see little need to learn more than the

mathematics needed to solve the straightforward problems encountered regularly outside of school.

Many educators mistakenly equate the use of concrete materials (or what is commonly referred to as "practical work") with realistic problem solving. Desforges and Cockburn (1987) in their study of classroom practice make the following observation:

Whilst practical work was evident, it played, by and large, only a supporting role to paper and pencil work. The children very rarely tackled **real problems** [bold emphasis is mine] in their mathematics or applied their thoughts to anything other than the procedures demanded by the worksheets. (p. 79)

Despite the claims about the practicality of mathematics and its relevance to the tasks and problems of the everyday world, young children do not necessarily share this view that mathematics is useful. Desforge and Cockburn found that children provided no references to the use of mathematics in their lives or the lives of adults, and many children cast doubt on whether adults indulge in mathematical activities (p. 100). Instead, in responding to questions about the purpose of mathematics, the majority of children believed that mathematics was studied because it was "their work." The following response typifies the view of the children interviewed in the Desforges and Cockburn study (p. 100):

Interviewer: Why do you measure things?

Kerry: For the book.

Interviewer: Why do you think it's in the book?

Kerry: It's our work.

For most young children following a traditional mathematics program, the applications of mathematics are relatively unimportant and, judging from the following spontaneous comments made by children in the Desforges and Cockburn study, the main object of school mathematics is to finish it quickly (p. 101):

I did eight pages today and I got them all right.

I've only got six more pages to go.

I started there and I've done this page, and this page and now I'm on this page and I've got only one, two, three, four, five more pages to go.

I've got to get on so that I can get on to book six.

I've skipped lots of books, y'know.

Considering all these difficulties, how can the mathematics program prepare students for problem solving in the real world? One industrial mathematician, in recently summarizing what industry expects of its employees, provides some guidance. He believes that mathematics education should develop students who possess the following traits by the time they leave school (NCTM, 1987, p. 10):

1. The ability to set up problems with the appropriate operations;
2. A knowledge of a variety of techniques to approach and work on problems;
3. The ability to work with others on problems;
4. The ability to see the applicability of mathematical ideas to common and complex problems;
5. The ability to handle and the willingness to accept open problem situations, since most real problems are not well formulated;
6. A belief in the utility and value of mathematics.

If students are to develop these traits and leave school with a better understanding of how mathematics is connected to the physical world, less emphasis should be placed on learning "recipes" to solve a narrow range of problems where all the information needed for a solution is conveniently provided, and the objective is to reproduce some recently learned technique. Instead, we need to give students many opportunities to work with problems that are as realistic as possible. Realistic problems or situations share many of the following characteristics:

1. They are not well defined and require some further specification and refinement.
2. They often have multiple solutions.
3. They require the collection of information.
4. They involve collaboration with other people.
5. They cannot be solved in a few minutes.
6. They involve some personal commitment.

Teachers need to be provided with concrete ideas and materials to get them started in realistic problem solving, or what some mathematics educators refer to as "mathematical modelling." The following example illustrates a type of problem that needs to be included in a revised mathematics program. The accompanying discussion shows how this problem differs from the supposed application problems presented in texts:

Here are five different brands of paper towels. Find which brand of towel is the best buy.

This problem, which is appropriate for middle years students, has a number of qualities that distinguish it from traditional problems. It is ill-defined and further elaboration is necessary before a solution can be determined. Questions like the following must be posed and answered before planning an approach: (1) What is meant by best buy?; (2) Does best buy refer to price, price per length, price per area, ability to clean, and so on?; (3) What materials do we need and how can they be obtained?

In a classroom where this problem was presented, one group of students considered the major function of paper towels, their ability to soak up spilled liquids, and proceeded to consider ways to assess, what they termed, the "soakability" of each brand of towel. This problem forced the students to select and carry out measurements and to consider how all the measurements

can be carried out in a consistent manner (i.e., controlling relevant variables).

This problem situation encourages student collaboration and facilitates group discussions. Students usually find that the most efficient approach is for each member of a group to carry out some related sub-task. They also must learn how to present their ideas effectively and how to react to the ideas presented by other group members. Realistic problems provide a medium to promote the important aspects of human communication: listening, reading, speaking, and writing.

Although, at first glance, the problem appears to be inconsequential, it usually generates considerable student activity. The students must gather information, make measurements, make calculations, organize and represent quantitative information, make judgments, and develop a conclusion.

Unlike a typical textbook problem, there may be several "right" answers produced by various groups depending upon how the problem is formulated. (A textbook would simply list the prices and length of each brand, expecting students to calculate a unit price.) With the paper towel problem, students begin to realize the importance

of examining the assumptions underlying any judgement or conclusion based on empirical methods.

The problem requires time to produce a solution. It cannot be solved in a few minutes and may require more than one class period. Teachers have to be well prepared and organized when they present students with this type of problem. They also find that their role changes to that of a facilitator: asking probing questions, providing materials, organizing groups, discussing ideas, and helping students overcome conceptual and arithmetical difficulties.

The Shell Centre for Mathematical Education located at the University of Nottingham has recently developed classroom materials designed to promote the skills in mathematical modelling. The inexpensive booklet *Beginning to Tackle Real Problems* is a 6-12 lesson "starter pack" for introducing some mathematics modelling into the mathematics curriculum. The authors have used the activities in the booklet in its present form for children over the age of 14. However, teachers can tailor the activities to different ages and abilities by varying the level of sophistication of the problems and of the answers expected.

The authors claim that their suggestions are based on research studies of how pupils, with no previous training, tackle real problems. The main purpose of the booklet is to provide teachers with enough experience and guidance to learn eventually to develop their own mathematical modelling activities. The booklet also provides teachers with some practical advice on the teaching of mathematical modelling.

Problems like the following are included:

1. The children in a class have complained that they cannot easily see the blackboard because the heads of the children in front block their views. Someone has suggested that a stage at the front of the classroom should be built for the teacher to stand on. The blackboard could then be placed higher up on the wall. Is this a good suggestion? (Diagrams of the side view and top view of the classroom are provided.)

2. Merchant Ltd. are planning to construct a customer car park on the vacant land next to their supermarket (a diagram is provided). If the land measures 54.2 m by 158.9 m, design the best layout for the carpark.

For each problem, the authors suggest ways for teachers to develop the following important modelling subskills:

1. Generating variables

2. Selecting relationships
3. Generating relationships
4. Identifying specific questions
5. Modelling objects and situations
6. Estimating
7. Validating relationships.

Numeracy through Problem Solving, also developed by the Shell Centre for Mathematical Education, is a comprehensive program to help students in the age range of 13 to 16 tackle real problems more effectively. Currently, the program consists of the following 5 modules:

- (1) Design a Board Game - in which students design and produce a board game that can be played and evaluated by other members of the class.
- (2) Produce a Quiz Show - in which students devise, schedule, run and evaluate their own classroom quiz.
- (3) Plan a Trip - in which students plan and undertake one or more class trips, and possibly some small group trips.
- (4) Be a Paper Engineer - in which students design, make and evaluate 3-dimensional paper products, such as pop-up cards, envelopes and gift boxes.
- (5) Be a Shrewd Chooser - in which students research and provide expert consumer advice for clients in their class.

Each module is designed to be completed within 10 to 20 hours spread over 3 to 6 weeks. A feature of each module is the importance attached to students working in groups, explaining their ideas and listening to each other, making their own decisions and living with the consequences, reflecting on their experience and learning from it, just as they do in life outside of the classroom.

Although the focus is on numeracy - the deployment of mathematical skills and reasoning in real problem solving - the authors have designed the modules to have a broad cross-curricular aspect. Skills from other curriculum areas, such as language arts, are inevitably called upon, as these are necessary for the presentation of the reasoned arguments essential for real problem solving. Since real world problem solving is usually spontaneous, individualized, and often involves the use of non-

routine procedures, schools can provide children with only an approximation of realistic problems. In a revised program, teachers need to be provided with a list of real-life, naturally occurring problems that can be understood and solved by elementary, middle, and secondary students. The emphasis here should be more on the collecting of information relevant to the problem and the development of a variety of problem solving approaches than on solving a given problem in a particular way considered correct by the teacher. In other words, students need to be encouraged to simulate the approaches likely to be taken by various problem solvers outside the classroom.

Teachers should also be provided with suggestions on how to develop and take advantage of problems occurring spontaneously inside and outside the classroom. As with other types of problem solving, realistic problem solving is not just a topic in a program; it is a perspective from which all topics in the mathematics program can be viewed. Teachers need to continually ask themselves this question: Can I provide children with situations where this mathematical concept would be seen or with problems where this mathematical procedure would be applied?

Footnotes

- ¹ These problems are often termed "word problems," "story problems," or "verbal problems." The term "word problem" seems oxymoronic; it is difficult to imagine how any problem can be expressed without using words. The term "story problem" is too vague to be useful. The term "verbal problem" suffers from the same deficiency as the term "word problem." Consequently, the term "translation problem" - translating written or verbal statements into mathematical expressions and then performing an algorithm - seems preferable to any of these other commonly seen terms.
- ² Several students wrote comments like, "How can I do this problem when I've only been taught metric units?"
- ³ I observed an adult who was so upset about the fractional distance represented in the problem, she spent most of her solution time trying to convert $2/5$ mile into a whole number of feet.
- ⁴ I should make it clear, if it is not obvious to the reader, that these terms are not intended to be given to children. We don't want teachers to be telling students something like: "To solve Part-Part-Whole problems, we first..." They are intended to aid teachers and curriculum developers in planning instruction.

- ⁵ It should be noted that most pre-schoolers have some rudimentary idea of the division process and can apply these two models to some division problems. For evidence of this, simply ask two youngsters to share a quantity of candies (partitive process) or to determine the number of pairs of socks given a collection of socks (quotitive division).
- ⁶ Many of these problems are variations of problems taken from *Problem Solving: A Handbook for Teachers* by Krulik and Rudnick.

CHAPTER 7

FOUNDATIONS OF NUMBER

In a revised elementary and middle years mathematics program, **a fundamental goal needs to be the learning about number and quantity and the development of a keen number sense.** Students must be expected to learn common number relationships and patterns, the meaning of the four operations (addition, subtraction, multiplication and division), and the terminology and symbolism associated with number and number transformations.

Part-Whole Relationships

Part-whole relationships are important ideas usually not identified as goals in most programs. Without an understanding of these ideas, students experience difficulties in learning mathematics. In reviewing the work of cognitive psychologists such as Lauren Resnick, several intuitive notions involving the composition of quantity and part-whole relationships can be identified:

1. all whole number quantities are compositions of other smaller quantities.
2. adding a quantity to another quantity increases the total quantity.
3. removing one quantity from another decreases the total quantity.
4. partitioning a quantity into several quantities does not change the total quantity.
5. one and only one number can be assigned to represent a quantity.
6. not all numbers represent quantities: e.g., numbers used to order or label objects.
7. two quantities can be made equivalent either by removing an appropriate quantity from the larger quantity or by adding an appropriate quantity to the smaller quantity.
8. the number assigned to represent a quantity is independent of the physical characteristics of the individual objects composing that quantity or the configuration of the entire quantity.
9. quantities can be ordered according to the numbers assigned to represent each quantity.

10. in a quantity partitioned into two sub-quantities, either sub-quantity can be determined given the value of the remaining sub-quantity and the total quantity.
11. quantities and their transformations can be represented by standard mathematical symbols and expressed in standard mathematical language.
12. mathematical symbols can be operated on without reference to any quantities.

The program should ensure that children eventually see these relationships as self-evident or intuitive through regular involvement in practical quantitative situations. For example, the notion that removing objects from a set reduces the quantity can be developed by having children form a set, remove some objects, and count the objects again.

Some evidence exists that direct instruction can promote these understandings, but not as easily or quickly as might be expected. Resnick (1986) studied children's strategies for equalizing sets when given verbal problems of the form, "If you have six marbles and I have four, what could we do to each collection to have the same number?" (p. 171). David, a 6-year-old subject in the study, used two strategies to equalize sets: by (1) diminishing the larger set or increasing the smaller set by the difference between the two sets (referred to by David as buy/sell); or by (2) combining the sets and sharing each half.

The researchers demonstrated to David a third and more complex strategy referred to as transfer: viz., transferring objects from one set to another to make them both equivalent. David was unable to learn immediately from this instruction even though he understood the purpose behind the transfer approach. After four months he could use this strategy to equate sets with a difference of 2 by transferring one object from the larger to the smaller set. It was not until 10 months after the initial instruction, however, that he could use the transfer strategy for virtually any sets and explain his procedure (p. 173). Resnick argues that the child himself needed a significant amount of mental construction in mastering this particular strategy. Resnick also found that the proportion of children who chose this more complex equalizing strategy increased gradually across grades as follows: grade 1 (0%), grade 2 (7%), grade 3 (11%), grade 4 (50%), and grade 5 (71%). Her results suggest that the majority of children will not be able to develop the concept of "average" (finding the middle quantity of two given quantities) until the later elementary or middle years.

Counting

Primary teachers know the importance of providing children with a good understanding of counting and its many applications. Recent research by mathematics educators and cognitive psychologists has demonstrated that counting is a pervasive

technique that many children depend on throughout their elementary years. Researchers also report that counting is a more complex process than is often supposed. A revised program should reflect this research into counting and ensure that counting skills are included in each elementary grade and maintained in the middle grades.

In the early primary grades, the initial emphasis should be on learning to count out quantities up to 20 objects: i.e., determining the cardinality of sets. Students surmount a major cognitive obstacle when they realize that the counting words and symbols follow a regular repetitive pattern: specifically, starting with a word representing a specific multiple of ten, each counting word represents one more than the previous counting word until another multiple of ten has been encountered (e.g., twenty (2 tens), twenty-one (2 tens, 1 more), twenty-two (2 tens, 2 more), ... twenty-nine (2 tens, 9 more), thirty (3 tens)).

Although most young children realize that a counting pattern exists, they often substitute their own creations for the correct number words in their initial attempts to replicate this pattern: e.g., substituting the word "five-teen" for fifteen. Once the counting pattern is recognized by children, counting becomes relatively easier to master and the remaining task is to learn the conventional terms for other multiples of powers of 10 (one hundred, three thousand, etc.). The research suggests that children first acquire some rote understanding likely through imitation (as most of us have learned to recite the alphabet, for example) and then use a discovered pattern to extend the count (Baroody, 1987, p. 85). Consequently, teachers need to be aware that children need explicit instruction to help them identify counting patterns as well as practice in stating and writing the counting words in order.

In the later elementary and throughout the middle grades, students should continue with counting but learn to skip count in large multiples (e.g., skip counting by 25s: 25, 50, 75, by 100s: 100, 200, 300, etc.) and eventually in non-whole number units (e.g., 0.5, 1.0, 1.5, 2.0, etc.; $1/4$, $1/2$, $3/4$, 1, $1\frac{1}{4}$, etc.).

The program should emphasise counting for a purpose - determining quantities and solving simple problems by counting and relating counting to the operations of adding, subtracting, multiplying or dividing. Middle year students should learn to use counting to estimate operations on fractions and decimals: for example, reasoning that 5 half cups of flour is equivalent to $2\frac{1}{2}$ cups by counting in increments of $1/2$ cup.

Young children need activities that reinforce important counting principles and promote their efficient use. For example, children need guidance in order to learn a systematic procedure for counting a large quantity of objects. Learning to coordinate the counting words while counting objects is a competence that only gradually becomes automatic. When counting large quantities arranged in a disorderly fashion,

children often encounter difficulties in keeping track of the counted and uncounted items. Children need to learn to count quantities using systematic approaches such as the following: (1) moving in a consistent direction (left to right, top to bottom); (2) separating counted from uncounted objects (placing the counted objects into a container, for example); (3) grouping objects into orderly arrangements such as pairs, or into logically connected collections such as considering every four quarters as equivalent to a dollar, for example.

Children do not understand initially the purpose of counting and do not realize that the last counted object determines the number in a collection of objects. Baroody (1987) recounts his interview with a 3-year-old who counted 4 stars in the proper fashion ("one, two, three, four") but, when asked how many stars were counted, she responded by counting the stars again. Baroody argues that, unless children understand the purpose of counting, questions probing an understanding of "how many" are pointless.

Baroody also states that children often find it difficult to produce quantities when following imperatives such as "Get me three pencils" or "Show me eight crayons." Although most kindergarten children can produce quantities upon request, they do so for only very small quantities. Baroody identifies a number of cognitive processes involved in producing quantities: (1) noting and remembering the requested number of items, (2) counting out each item correctly, and (3) monitoring and stopping the counting-out procedure.

Many children continue to count quantities one object at a time even when they know how to skip count orally. Several years ago, I interviewed a 10-year-old girl having difficulties in learning mathematics. I first determined that she could easily count aloud by tens to 100: Ten, twenty, thirty, and so on. She also correctly wrote the missing numbers in the sequence: 10, 20, 30, __, __. I asked her to determine the number of sticks in a collection of popsicle sticks bundled in groups of ten. Even though I emphasized that each bundle contained 10 sticks, she proceeded to remove the elastic band and count each stick one by one. As a general rule, we need to ensure that children see skip counting as an efficient counting procedure rather than as a parrot-like recitation.

The following partial list of competencies and understandings related to the counting process should be incorporated into a revised program:

1. Orally reciting and writing the number words in their correct sequence from 1 to some established limit.
2. Orally reciting and writing the number words in their correct sequence from a beginning other than 1 to some established limit.

3. Orally reciting and writing the number words in their correct sequence by multiples of some number to some established limit.
4. Orally reciting and writing the number words in their correct sequence by multiples of some number to some established limit beginning from some given number.
5. Performing competencies 1 to 4 in reverse order.
6. Understanding and using the terms 'before,' 'after', and 'between' in reference to counting.
7. Applying counting in an efficient manner to determine quantity, and to compare two or more quantities.
8. Recognizing quantities of up to 5 by visual inspection rather than counting each item and using this recognition to count out efficiently larger multiples of the recognized set.
9. Knowing the relationship between counting and the operations of adding, subtracting, multiplying and dividing.
10. Using counting to determine the placement of an object in an ordered collection: i.e. using ordinal numbers.

Place Value

A high priority in a revised program should be to ensure that all elementary and middle years students develop a good understanding of the conventional place value numeration system. Children's understanding of our ingeniously compact but necessarily abstract numeration system develops slowly. To be considered competent in handling our numeration system, the following properties and conventions must be understood:

1. Positional property. The position of each digit in a multi-digit numeral determines the represented quantity.
2. Base ten property. The place value positions from right to left represent increasing powers of 10.
3. Multiplicative property. Each digit in a multi-digit numeral represents the product of the digit and the power of 10 assigned to the digit's position.
4. Additive property. The quantity represented by a multi-digit numeral is determined by adding the multiplicative values represented by each individual digit.

5. Special name convention. Special names are given to the positive and negative powers of one thousand: viz., thousand, million, billion, thousandth, millionth, billionth.

6. Reading convention. The number of digits determines the conventional reading of a multi-digit numeral. If the number of digits is equal to or less than 4, the numeral is read from left to right expressing the value represented by each digit (e.g., 1234 is read "1 thousand, 2 hundred, thirty, four). If the number of digits is greater than 4, groups of three digits are formed from right to left by placing a space to separate each grouping.¹ The number is read from left to right expressing the value represented by each digit in a 3-digit grouping followed by the special name of the grouping (e.g., 12 345 678 is read as "12 million, 3 hundred-forty-five thousand, 6 hundred-seventy-eight").

7. Decimal point convention. To distinguish between digits representing the negative powers of ten and the non-negative powers of 10, a decimal point is placed between the unit digit and the tenths digit. All place value properties and conventions apply to multi-digit decimal numerals. The decimal point is usually read as "and," "decimal," and often "point."

Even by the end of the fifth grade, some children have not learned the most basic property, the base ten property. Ross (1989) studied grade 2 to 5 children's understanding of two-digit whole number place value notions. Most students were able to count correctly a quantity of 25 objects, but less than half knew that the digit 2 represented 2 tens. Students gave a variety of incorrect responses: (1) not relating the digits to the counted quantity; (2) stating an incorrect relationship - the 5 represents 'half of 10' or the 2 represents 'count by twos'; (3) relating the digits to their face values - the 2 represents 2 sticks and the 5 represents 5 sticks.

The students were much more successful when the two-digit numerals were represented by concrete objects arranged to make the base 10 grouping more obvious (e.g., base 10 blocks). Ross reported that students found it extremely difficult to name the numbers represented by non-standard arrangements of the place value objects. Only one third of all the students realized that the numeral 52 could be used to represent four ten-blocks and twelve unit cubes, for example.

Ross also found that children tend to confuse the face value of each digit with its place value. In her study, children were asked to count out 26 toy wheels and write the numeral representing this quantity. They were then asked to sort the objects into groups of four to represent the sets of wheels needed to make toy cars (i.e., 6 fours and 2 ones). When the children correctly formed a partitioned collection of 6 groups of 4 wheels and 2 left over, the interviewer encircled each digit in the numeral 26 written by the child and then proceeded to ask the child to explain the relationship

between each digit and the resultant partitioning. Nearly half of the third graders incorrectly responded that the digit 2 in the written numeral stood for two of the objects, while the digit 6 stood for the six groups or six cars. Ross provides this cautionary advice to teachers: "Students in their early stages of understanding of place value appear to understand more than they actually do" (p. 49).

Based on a review of the literature and the findings of her study, Ross proposed that the understanding of two-digit numerals of elementary school children proceeds through five stages:

Stage 1: whole numeral. Children at this stage of development can count quantities up to 99, name them and possibly even write the corresponding numerals. But they view quantities as whole amounts and do not ascribe meaning to the individual digits or number names.

Stage 2: positional property. Children at this stage know that the right digit in a 2-digit numeral represents the number of units and the left digit represents the number of tens. But they cannot name the quantities represented by these digits.

Stage 3: face value. Students at this stage interpret each digit as representing the number indicated by its face value. The digits correspond to any grouping of this size rather than to any group of ten. They do not realize that the number represented by the tens digit is a multiple of 10.

Stage 4: construction zone. At this stage, students know that the left digit represents collections of ten object and that the right digit represents the remaining single objects. However, this knowledge is characterized by unreliable task performance.

Stage 5: understanding. Students at this stage understand that the individual digits represent a partitioning of tens and ones. They also can determine the quantity for collections that have been partitioned in unconventional ways.

Ross found that only about a quarter of the grade 2 to 5 sample of children were at stage 5 in their understanding of two-digit numerals. At the grade 5 level, about half of the students had reached stage 5, while the other half had reached stage 4. Assuming that the findings of the Ross study are applicable to Saskatchewan, the **current expectations regarding place value understanding seem unrealistically high.** Consider the current standards of whole number place value for grade five, for example:

1. reads and writes numerals up to a billion.
2. determines the place value of each digit given a numeral with nine or less digits.

3. compares two nine-digit numerals.
4. converts standard numerals to expanded numerals and expanded numerals to standard numerals using the format $756\ 254 = (7 \times 100\ 000) + (5 \times 10\ 000) + (6 \times 1000) + (2 \times 100) + (5 \times 10) + (4 \times 1)$.
5. rounds off a given number to the nearest thousands place.

In a revised curriculum, the expectations for place value understanding should be more realistic. I recommend that **whole number place value work in the elementary grades be restricted to numbers less than 10 000 and decimal work be restricted to thousandths**. These limits have not been not chosen arbitrarily. The large majority of numbers seen in society are well within the proposed range. Rarely will elementary school children be exposed to numbers larger than 10 000 and, because most decimals are seen in the context of money and measurement, a working knowledge of three decimal places is adequate to solve almost all practical problems. The middle years program should include work with larger and smaller numbers than those proposed and introduce students to exponential notation, but only in the context of problem solving with a scientific calculator.

Although extensive experience with embodiments such as base-ten blocks, abaci, and other appropriate manipulatives should be provided, they need to be used cautiously. Ross argues that, when the place value materials make the base ten relationship explicit, a stage 3 (face value) interpretation can be reinforced. As Ross suggests, "With these materials, the teacher and manufacturer may have 'embodied the ten,' but the student need not" (p. 50).

Ross suggests that teachers provide children with problem-solving tasks that require them to think about useful ways to partition and compose numbers: for example, arranging a sum of 52 cents in several ways - 5 dimes, 2 pennies; 4 dimes, 12 pennies, etc. Children also need to learn to use their knowledge of place value to simplify calculations: for example, calculating 25 and 35 by adding 20 and 30 and then 5 and 5. Regular practice in mentally calculating dictated numbers should be provided: e.g., adding the following numbers mentally, "Two hundred, one hundred, one thousand, fifty eight."

Middle years students should learn how to order decimals, locate their relative position on a number line, and, most importantly, transform them into other equivalent forms: for example, 0.123 can be thought of as 12 100ths + 3 1000ths, as slightly greater than 12 100ths, as about 12%, as less than $1/2$, as slightly less than $1/8$, and so forth. Students should learn these types of transformations while engaged in work with suitable concrete materials: base 10 blocks, money, measuring materials, place value charts, abaci, and so on. We also need to ensure that more oral work is

included to offset the excessive amount of symbolic work presented in texts and workbooks. Students who can calculate and write out large numbers cannot necessarily read them. A common error is to record orally presented numbers in parts. For example, when students are dictated the number "one thousand, two hundred, five," they typically write 10002005.

The calculator provides a useful medium for promoting the reading and saying of numbers. Teachers can dictate several large numbers and ask students to calculate with them: e.g., "use your calculator to divide 'one thousand, four hundred, forty four' by 'twelve' and read me the answer to the second decimal place." This activity is practical only if students are allowed to use a calculator; otherwise the production of a calculation rather than the naming and interpreting of numbers becomes the dominant activity.

Fractions

Except for the most rudimentary ideas, very little fractional work should be included in the elementary program, certainly less than that included in the current program and contained in most elementary texts. One reason to keep work with fractions to a minimum in the elementary years is that, other than simple usage (quarter to 10, half dozen, half dollar, etc.), fractions are rarely seen outside of school. Everyday situations requiring that fractions be added or subtracted are difficult to find and we cannot continue to justify the teaching of routines to determine equivalence, sums, differences, products and quotients to children in the elementary years.² The majority of operational work on fractions should be delayed to the middle years and instruction on fractions should continue in grade 9 and likely into the secondary years.

Children generally encounter difficulties in learning to work with fractions. Hence, the inherent difficulty of fractions is another reason to keep their study to the barest minimum in the elementary years. For example, NAEP (National Assessment of Educational Progress, 1979, p. 3) found that even some 13-year-olds were still struggling to understand the concept of a fraction: e.g., 28% could not name the fractional part of a set when shown a collection of blue and white marbles. Children's difficulties with fractions have been reported in several countries.³

Fractional work in the elementary years should be restricted to the development of understanding of the various meanings and applications of fractions. The goal should be to help children understand that fractions are another type of number, to make them familiar with their practical uses, and to introduce them to the associated symbolism and terminology. At the elementary level, there is no need to teach the formal methods of computing, including the production of equivalent fractions. Such

work should be developed informally in the context of practical problems: e.g., noting one half cup of water is equivalent to a two quarter cup, and adding a further one quarter cup produces a three quarter cup.

What meanings should be introduced in the elementary and middle grades? In their review of the literature on students' difficulties in learning fractions, Hope and Owen (1987) identified five meanings commonly ascribed to fractions. First, fractions can be considered as **measures** whenever an object is partitioned equally into smaller units: quantities of objects are partitioned into smaller but equal-sized quantities; lengths are partitioned into smaller but equal linear units; areas or regions are partitioned into smaller and usually congruent areas; solids are partitioned into smaller and usually congruent volumes. The major work in the elementary grades involving fractions should be devoted to developing a good understanding of this meaning.

To become familiar with fractions as measure numbers, children need to acquire a number of understandings as well as some accompanying technical competencies: specifically, (1) knowing how to partition an object or quantity into equal-sized or congruent parts, and (2) knowing how to represent the partitioning using the symbols for a fraction and the associated terminology.

Children acquire the partitioning competency only after many concrete experiences. Partitioning can be difficult to perform because it involves a number of cognitive processes: (1) it requires the child to be a conserver of number and quantity;⁴ (2) it requires some technical proficiency (e.g., knowing how to divide a region or line segment into equal parts; (3) it varies from setting to setting (i.e., knowing how to subdivide a number of objects into equal sets does not necessarily help in the partitioning of a number line into equal segments.

Not surprisingly, studies have should shown that students find partitioning difficult to perform. Peck and Jencks (1981) found that over 50% of grade 6 students when asked to draw sketches for various diagrams drew diagrams (usually a circle) subdivided into a number of non-congruent parts. Payne (1984) found that only 31% of 13-year-olds could place the fraction $\frac{3}{4}$ on a number line on which the points 0 and 1 were labelled.

Children who work with fractional parts of regions often ignore the requirement that each part be equivalent in area. The following excerpt of an interview (Clay & Kolb, 1983, p. 245) with a grade 4 child named Brian illustrates this common difficulty, which can persist for years:

Interviewer: Do you know this symbol? (Writes " $\frac{1}{3}$ ")

Brian: Yes, that's one-third.

Interviewer: What is one-third?

Brian: Well, I can show you one-third. [Draws a circle divided into 3 equal regions.]

Interviewer: [Draws a circle divided into 3 unequal parts.] Is this one-third?

Brian: Yeh, that's one-third.

Interviewer: Why?

Brian: Because it's one of three parts.

Brian was later asked why he always tried to make the parts equal when he drew a representation of a fraction. He replied, "Well, that's the way my teacher does it." When asked why he thought her teacher drew equal parts, he said, "Well, I guess because it's easier or it looks better, but, ... um .. maybe the parts don't have to be the same."

In order to represent a partitioning with the correct symbolic notation, children must learn that the denominator denotes the number of partitions and the numerator denotes the designated number of parts. This appears to be a straightforward concept for most students until they encounter "improper fractions" in which the numerator is larger than a denominator. Many children see such situations as a paradox: How can there be a larger number of smaller units than the entire whole? Chaffe-Stengel and Noddings (1982) found that fifth and sixth graders when presented with problems involving imagined pizzas, candy bars and pies were reluctant to consider improper fractions because they claimed that, "The bigger one can't be over the smaller one." Historically, many 16th century writers refused to use the symbol a/b to represent numbers larger than 1.

The major emphasis in the elementary grades should be to expose children to the measure interpretation of fractions and to introduce the symbolic notation. Furthermore, children should learn to appreciate the relative size of fractions through involvement in the following types of activities:

1. Representing fractions using a variety of materials: number lines, various geometric shapes and solids, paper folding, and so forth.
2. Generating equivalent fractions.
3. Comparing fractions with like denominators.

4. Comparing fractions with like numerators.
5. Ordering fractions with respect to other fractions, especially $1/2$ and 1 : e.g., seeing that $3/10$ is less than $1/2$ and 1 , $2/3$ is greater than $1/2$, and concluding that $3/10$ is less than $2/3$.
6. Adding and subtracting fractions with like denominators using materials.
7. Estimating the sum of two or more fractions and the difference of two fractions by using a comparison: e.g., recognizing that the sum of $1/2$ and $3/4$ is larger than 1 because $1/2$ and $1/2$ equal 1 .

The remainder of the meanings of a fraction are more suited to the middle years program than the elementary program.

Quotients are a second meaning attributed to fractions. Formally, in this meaning, fractions are considered solutions to equations of the form $bx = a$, where a and b are integers and $b \neq 0$. Informally, a fraction represents a division sentence in which the numerator is a dividend and the denominator is a divisor. Studies have shown that children do not grasp this meaning easily. For example, Hart (1982) found that many British middle years students refused to believe that the expression 16 divided by 20 can be expressed as a fraction. Fractions as quotients should be postponed until the middle years.

Third, fractions can be considered as **ratio** numbers - hence the term **rational** numbers: e.g., 1 can frozen orange juice for every 3 cans of water can be expressed as either $1:3$ or $1/3$. Ratio and proportion is a very difficult notion for children to comprehend even for secondary students.⁵ This meaning can be introduced informally in the elementary grades through practical problems (e.g., how many cans of water are needed if 10 cans of orange juice are used?), but the symbol a/b should be postponed until the middle years.

Fractions can be viewed as **operator** or **mapping** numbers. For example, a $1/8$ operator can be viewed as the mathematical model of a machine that assigns eight sticks of gum to every one package. This view can be introduced informally using the machine analogy, but again studies have shown that the major work should be postponed until the middle years.

Ratio and Proportion

The importance of ratio and proportion becomes evident in their abundant applications: in daily use (comparison shopping, recipe conversions, working with speeds and rates), in measurement (changing between measurement units such as from cm

to m), in scales and scaling, in other mathematical areas (probability, geometry, graphing), and in middle years school subjects such as science, health and social studies.

Although proportional reasoning is "generally regarded as one of the important components of formal thought acquired in adolescence" (Hoffer, 1988), it develops slowly and, generally speaking, is usually poorly understood even by students taking secondary mathematics courses. The most widely reported research into students' understanding of ratio and proportion involves, what is commonly referred to as, the "Paper Clip Task" or, alternatively, the "Mr. Short/Mr. Tall Task."

Subjects are presented the height of Mr. Short measured in paper clips (typically 6) and in buttons (4) and asked to explain how they can find the height of Mr. Tall in paper clips given his height in buttons (6); in other words, the solution requires proportional reasoning. Nearly one third of grade 8 students determined a solution by focussing on differences instead of proportions, as typified by the following response "Mr. Tall is eight paper clips high. He is two buttons higher than Mr. Short, so I figured he is two paper clips higher" (Karplus, Karplus, & Wollman, 1974).

Most studies indicate that students find it relatively easy to apply proportional reasoning to problems employing integral ratios such as 2:1 but much more difficult to use non-integral ratios such as 3:2. Researchers using other experimental tasks to investigate proportional reasoning (similar triangles, lever-arm balance, mixing lemonade) report similar findings about student reasoning and difficulties.

Karplus, Pulos, and Stage (1983) found that student reasoning when presented with a proportional problem can typically be placed into one of the following four categories:

Category 1: Those responses that seem incomplete or illogical - don't know, guess, inappropriate quantitative operations.

Category 2: Those responses involving a qualitative comparison of four given amounts using more, less, or equivalent terms.

Category 3: Those responses involving the computation of differences or remainders.

Category 4: Those responses involving the computation and comparison of proportional relationships, possibly with arithmetic errors. (p. 54).

Karplus et al. maintain that their research and those of other researchers demonstrate conclusively that traditional procedures requiring students to form proportions and then cross-multiply to solve "problems" is not well understood by early adolescents. They cite a study that "found that, even among eleventh-grade students, fewer than 15% used such strategies"

(p. 79). The results of grade 10 students on the following item included in the 1985 B.C. Mathematics Assessment study suggests that many students possess only a partially formed understanding of cross-multiplication:

A map of a large ranch was drawn to a scale of 2 cm to 50 m. On the map the distance from the house to one of the barns was 13 cm. What was the actual distance between the house and the barn? (Robitaille & O'Shea, 1985, p. 353)

Less than half chose the correct response 325 m. The number of students choosing the following distractors appears to indicate great confusion about cross-multiplication: 13% chose 650 m (13×50), 10% chose 65 m (13×5), and 11% chose 1300 m ($13 \times 50 \times 2$). Despite poor student performance, teachers and textbook authors continue to present this strategy to students, even to those in the early middle years.

What can middle years teachers do to help students develop an understanding of ratio and proportion? As a first step, they should lower their expectations regarding student mastery of the computational techniques normally associated with solving ratio and proportion problems. Teachers should try merely to ensure that students are well prepared conceptually. In particular, they should realize that the ability to carry out a technique such as cross-multiplying is not strong evidence of understanding. Secondly, students should be provided with activities that emphasise the concepts underlying ratios and proportions, particularly, variables and their relationships (Karplus et. al, 1983).

Hoffer's (1988) suggestions for teaching ratios reflects a sound conceptual approach to the teaching of ratio and proportion. He argues that students can benefit greatly from numerous verbal activities that stress the scaling up and down of familiar and common ratios: for example, those involving money, time, length, mass, capacity and other measurement relationships; those involving common units such as dozens, pairs, tires, fingers, and so forth. Tables of ratios such as the one below should be used:

1 nickel	5 pennies
2 nickels	10 pennies
3 nickels	15 pennies
4 nickels	? pennies

Hoffer emphasises that "teachers must help students learn to use multiplication facts directly without resorting to repeated addition" (p. 296). Students need to be aware

that not all ratios involve integral ratios and should be challenged with scaling up and down activities such as the one below:

$$1 \text{ m} = 100 \text{ cm}$$

$$2 \text{ m} = 200 \text{ cm}$$

$$3 \text{ m} = 300 \text{ cm}$$

$$4 \text{ m} = 400 \text{ cm}$$

Use the information above to answer the following questions. Be able to explain your method to your partner.

$$3.5 \text{ m} = ? \text{ cm}; 225 \text{ cm} = ? \text{ m}$$

Students should also learn that some ratios when applied to real world situations often make sense only when integral values are involved: for example, in calculating the number of nickels in 13 cents, an answer of 3.6 nickels is mathematically correct but nonsensical. Whenever possible, ratios should be placed in an everyday context to ensure that students can develop a sense of what is and what is not reasonable.

Students can then use their newly learned understanding to work informally with other common but less straightforward ratios, those involving consumer decisions for example.

If 3 tapes can be purchased for \$12.48, construct a table showing the cost of 6, 9, 12 and 15 tapes. Explain how this table can be used to determine the cost of any number of tapes from 1 to 15.

Once these preparatory activities are completed, more advanced techniques can be considered, specifically the unit method and the common multiple method.

The Unit Method

Scaling ratios to a single unit is a common real-world strategy that appears to be under-emphasised in the present program. Real world problems such as the following can be used to introduce the technique:

Which is the better buy, 5 pairs of socks for \$10.00 or 8 pairs of socks for \$15.00?

Students should learn that it is not always necessary to compute both ratios to determine the better buy. For example, in the preceding problem, the better buy can be determined by first figuring the easily calculated unit cost of one pair (\$10.00/5 pairs) and then using this information to determine the cost of 8 pairs, and finally comparing this price to the cost of the other pairs. Students should learn that consumers rarely apply the sophisticated mathematical techniques presented in school to solve practical problems involving ratio and proportion.

Students should also learn how to deal with ratios involving fractions, a variation of the unit method:

The following ingredients are need to make about 5 dozen oatmeal cookies:

3/4 cup shortening	1 cup brown sugar
1/2 cup granulated sugar	1 egg
1/4 cup water	1 teaspoon vanilla
1 cup all-purpose flour	1 teaspoon salt
1 teaspoon cinnamon	1/2 teaspoon soda
1/2 teaspoon cloves	1 cup raisins
1 cup chopped nuts	3 cups quick-cooking oats

If 3/4 cup shortening is used for every 1/2 cup of sugar, how much shortening is needed for every 3 cups of sugar? Explain your answer.

If a cook used 4 cups of raisins, figure the amount of each remaining ingredient that must be used?

Spreadsheets can be used to help students see relationships involving ratios. For example, students can see what happens to the amounts in the preceding recipe as the number of servings change.

The Common Multiple Method

As the term suggests, this method consists of determining a common multiple of one term of each ratio being considered. Hoffer provides this example:

Suppose pitcher A won 18 of 25 games for a particular season. Pitcher B won 15 of 20 games played for the same season. Which of these pitchers has the better record of pitching for the season?

Clearly, the ratios can be easily compared by finding a common multiple of 20 and 25 and then transforming each ratio accordingly. This method is frequently used to compare rates by converting them to percentages.

The common multiple method seems not to be well understood by students. In the 1985 B.C. Mathematics assessment, about 40% of grade 10 students did not correctly answer the following multi-step problem, which can be solved easily by finding a common multiple of 100 and 6500:

John and Jean are planning a vacation trip to Winnipeg next summer. They estimate they will travel a total of about 6500 km. Their car uses

11L/100 km. Approximately how many litres of gasoline will they need for the trip? (Robitaille & O'Shea, p. 122)

Their performance appears especially weak because the choices of answers were 60, 70, 600, and 700 and, consequently, the correct answer 700 can be determined rather easily by estimating rather than producing an exact calculation.

Percentage

Although percentage is commonly seen in the everyday world of the consumer and worker, students have historically found this special ratio difficult to understand. For example, in Schloring's study of 200 000 students in grades 5 to 12, 80% of 12th grade students could not compute 2.1% of 60. This study was conducted over 50 years ago in 1936. In 1981, 2890 people over 16 years of age were interviewed in England, Wales, and Scotland (Advisory Council for Adult and Continuing Education, 1982). They were asked to answer a number of supposedly relatively simple and straightforward questions. One question presented a sign on which the following advertisement was displayed: "25% off all marked prices." The adults were asked the following question: "If you saw a sign in a shop would you expect to pay a half, three quarters, a quarter, or a third of the original price?" About one third either answered the item incorrectly or provided an incomplete response such as "a quarter off the original price."

More recently, the 1985 B.C. Mathematics Assessment found that grade 10 students were only "marginally proficient" on items involving percentages. The results on the following item are typical (Robitaille & O'Shea, p. 122):

Linda's new bike cost \$159.99 and the sales tax was 5%. How much did she pay including tax?

<u>Choices</u>	<u>% of students</u>
A. \$164.99	11
B. \$167.99 (correct answer)	50
C. \$172.98	12
D. \$177.97	9
E. I don't know	12

What seems especially disturbing is that this item could be answered easily by using one's "loaf" instead of blindly resorting to a calculation: "The bike costs about \$160.00, 10% of 160 = 16, so 5% of 160 = 1/2 of 16 = 8, total is about \$168. Correct choice must be \$167.99."

In Saskatchewan, over 40 percent of grade 9 students in Giesbrecht's study (1977) could not correctly calculate 2% of 600 given the choices 1200, 300, 120, 30 and 12. About 20 percent chose the incorrect answer 300 (equating 2% with $1/2$, I suppose). Fortunately, performance improves throughout the secondary grades with only about 14 percent of grade 12 students unable to answer this question correctly.⁶

Student performance on items involving compound interest are especially low. Giesbrecht found that **only about 14 percent of grade 9 students** in Saskatchewan could correctly compute the total of a savings account in which the initial amount of \$200 is invested for 2 years compounded at a 5% annual rate of interest. (The term "compounded" was not used but the intent of the question was quite clear.) The most common answer \$220 indicated that students likely calculated 5% of \$200 for each of the two years and added the resulting \$20 to the principal of \$200.

Students also confuse fractions with ratios, and frequently set up a ratio to describe a situation without carefully examining its appropriateness. Students in the 1985 B.C. Mathematics Assessment were presented with the following question (p. 122):

At a party the ratio of boys to girls was 2 to 1. What percent of the people at the party were girls.

<u>Choices</u>	<u>% of students</u>
A. $66\frac{2}{3}\%$	10
B. 50%	29
C. $33\frac{1}{3}\%$ (correct answer)	43
D. 200%	9
E. I don't know	9

What can be done to improve student's understanding of percentage? First, percentage must be recognized as a difficult topic whose study should be included in each year of the middle years mathematics program and in the introductory years of the secondary mathematics program. While the current middle years program does include the study of percentage in grades 7 and 8, little work on percentage is included after grade 8.⁷

Second, the instructional emphasis in the middle years should be on developing understanding rather than computational facility. The current heavy emphasis on computation often leaves students without a clear understanding of the concept and with little skill in the mental manipulation of percents (Sobel & Maletsky, 1988). In a revised program, teachers need to make greater use of concrete aids, manipulative

experiences, visual imagery, and useful analogies to help students grasp this often abstract concept.

The 10 x 10 chart is an especially useful aid to visually represent percentage situations and to demonstrate the relationship between percentage and its relationship to other number representations such as decimals and fractions. By shading in the appropriate sections, students can see the relationship between the fraction $\frac{1}{4}$, the decimal 0.25, and the corresponding percent 25%. Students can also use the chart to reason out problems such as the following: (1) Find 20% of \$25; (2) Nine is 75% of what number?

Students should also learn to solve straightforward percent problems by referring to these commonly seen ratios: either thinking of a percent as cents per dollar, dimes per ten dollars, or dollars per 100 dollars. To solve the problem, "find 2% of \$15.00," the following reasoning should be encouraged: "2% means 2 cents for every 1 dollar, or $2 \times 15 = 30$ cents for 15 dollars."

They should also learn to interpret statements containing quantitative information expressed as a percent. The following types of statements probe student understanding and require no calculation:

Mr. Jones claims he is correct 100% of the time. Comment on his claim.

The chance of rain tomorrow is 10%. Should you find your umbrella.

Don Cherry claims that Bobby Orr, his choice for the greatest hockey player who ever lived, always gave a 110% effort when he played. Comment. Explain why Don Cherry talks like this.

A new diet reported in a supermarket tabloid claims that Ms. Roseanne Barge lost 50% of her weight in 6 months. Is this possible?

Third, students should be learn to reason out estimates in various ways as illustrated in the following examples:

Problem	Reasoning
1. Find the percent 15 is of 48.	15 out of 50 is equivalent to 30 out of 100. Answer is slightly more than 30%.

- | | |
|--|---|
| 2. Calculate 11% of \$250 | 10% equals $1/10$; $1/10$ of \$250 = \$25.
Answer is about \$25. More precisely,
1% can be thought of $1/10$ of 10%. Since
$1/10$ of \$25 = \$2.50, the exact answer
is \$27.50. |
| 3. If 80% of the regular price equals \$2.40, calculate the regular price. | 10% of the regular price must equal
$10/80$ or $1/8$ of \$2.40 or 30 cents.
Therefore, if 10% equals 30 cents,
100% equals \$3.00. |

As has been recommended throughout this report, teachers should encourage students to reason out problems in a variety of ways and to look for relationships that might aid in calculation: for example, using the knowledge that 10% is equivalent to $1/10$ to work out problems involving 15%, the traditional gratuity rate. To aid in this type of reasoning, students need to be able to recall various benchmark percents and their corresponding fraction and decimal equivalents: specifically, the percent and decimal forms of the common fractions having a denominator of 2, 3, 4, 5, 8, 10, 20, 25, and 50. Students should use these commonly seen relationships to solve practical problems: for example, estimating 35% of \$48.97 by figuring $1/3$ of \$48.00.

In summary, the number notions I have identified in this chapter form the basis for all learning in mathematics. It is apparent from a review of the research and from personal observations that many students do not have an adequate grounding in the foundations of number and, consequently, lack a good number sense. They may fool teachers by their ability to move symbols around on a piece of paper, but they flounder when confronted with a problem or situation requiring them to apply their knowledge of number and quantity.

Unfortunately, teachers continue to prefer facility with symbols over conceptual understanding. I suspect one reason that teachers do not provide students with enough concrete work and activities designed to stimulate "thinking" is because they also lack a deep understanding of number. This issue was raised at an NSF conference on number sense I participated in last year in San Diego. Zvia Markovits (1989) reported she was involved in a research project on number sense, number size, mental computation, and computational estimation. She decided to administer her number sense questionnaire to prospective elementary school teachers. Her questionnaire was similar to the one she used in her research with sixth and seventh graders. The student teachers told her that they found the questionnaire to be difficult, not because of the content but because they had to explain their answers. They said that being required to explain made them think, that they had never thought before about how they do problems, and, prior to this test, had never had to explain their answers.

I close this chapter with a few sample items, their results, and Zvia's insightful remark. Her items provide us with an idea about what should be included as the **real** goals of elementary and middle years mathematics.

- On placing the decimal point in 15.24×4.5 given 6858 (the answer less the decimal point), 79% said it should be after the six, 6.858.

- Thirty-one percent were unable to order 0.53, $14/13$, $5/12$, and 0.993, from smallest to largest.

- When asked if the answer to $264/0.79$ would be more than, equal to, or less than 264, 49% said it would be less, because you divide. But if asked to do the computation, all of them could find the correct answer.

- Only 39% said that there would be an infinite number of decimals between 0.46 and 0.47. Others said, "There is no room between."

- When asked if the answer to 52×28 would be more than, exactly, or less than 1500, 38% said it would be exactly 1500.

She writes, "Looking at these results makes one wonder, 'What have these students been doing for so many years in school?' Probably nothing related number sense" (p. 81).

Footnotes

¹ In the SI system, spaces are used instead of commas. The space is optional in 4-digit numerals.

² The present program includes the algorithms for adding and subtracting proper and mixed fractions in year 5. The program objectives for the same year includes the introduction of the algorithms for multiplication and division; however, students are expected to use informal rather than formal methods to calculate products and quotients. The prescribed text for year 5 does not include any work with multiplication and division of fractions.

³ Most early civilizations went to great lengths to avoid working with fractions. The commercial fractions of ancient Rome were based on the "as," originally a pound of copper, and Roman merchants eliminated the need for fractions by assigning names to fractional parts of the "as." Thus, business transactions involved "uncias" ($1/12$ of the "as"), "sex-tans" ($2/12$ of the "as"), "semuncias" ($1/24$ of the "as"), and so on.

- ⁴ Most students in the elementary grades and some in the middle grades are influenced by their perception and often think "one quarter" formed one way is not equal to "one quarter" formed another way.
- ⁵ Because of their importance, ratio, proportion, and percentage will be explored in greater detail in the next section of this chapter.
- ⁶ I suspect that this apparent increase in performance is due in part to the large proportion of students who drop out of secondary school, over one third of grade 8 students leave school before graduation.
- ⁷ The current Division III curriculum guide lists only one learning outcome for year 9: "applies the concept of percent to solve problems." Even this obviously vague objective is missing from the Saskatoon course outline for Grade Nine Algebra. The word "percent" isn't mentioned.

CHAPTER 8

BASIC NUMBER COMBINATIONS

In his 1962 short story *The Feeling of Power*, Isaac Asimov describes a future society where people have become completely dependent upon machines. People have lost the ability to solve even straightforward calculations such as recalling a product or sum. This futuristic world is startled to find that a low-grade technician has managed to reconstruct some of the ancient calculation methods usurped by machines. Shuman, the chief programmer of this advanced technological society, recognizes that this re-discovered knowledge may begin a process whereby human thought may be liberated from the intellectual servility imposed by machine:

Nine times seven, thought Shuman with deep satisfaction, is sixty three, and I don't need a computer to tell me so. The computer is in my own head. And it was amazing the feeling of power that it gave him.

As science fiction writers like Asimov correctly predicted, the trend of technology has been to devalue once prized memory and procedural skills; however, not all current school-learned knowledge will become obsolete. For example, many everyday problems requiring a calculation will continue to be handled by recalling a previously learned basic number combination rather than by turning to a machine. What self-respecting adult would want to be seen using a calculator, no matter how accessible, to carry out such practical tasks as figuring the number of nickels needed in an exchange for a quarter or determining the cost of 5 items priced at 30 cents each? Teachers can expect little support from the public if future generations of graduates have to depend upon a machine for every quantitative situation requiring a calculation, especially those that involve the recall of a basic number combination.

Elementary teachers have traditionally been expected to ensure that all children can recall instantly the so-called 390 basic number combinations for addition, subtraction, multiplication and division.¹ This knowledge is a special form of mental calculation and, hence, the following discussion could have been logically placed under that heading in the chapter on calculation. But a separate discussion is justified because teachers and parents consider the "recall of basic facts" to be an extremely important educational objective.

Unfortunately, some elementary teachers spend so much instructional time in developing mastery of the basic number combinations that other more important goals of mathematics education are neglected. In a revised program, we need to ensure that recall is developed in an economical manner without sacrificing a more com-

prehensive and more purposeful mathematics curriculum. This goal can be met in the following ways: (1) **by shifting some of the responsibility for developing memorization to middle years teachers**; (2) **by placing a greater emphasis on reasoning out the combinations**, especially during the early stages of learning, and (3) **by using effective practice procedures when memorization is considered an important instructional goal**.

Shifting Expectations of Memorization

An examination of curriculum guides and textbooks currently in use reveals that children are expected to memorize all addition and subtraction combinations by the end of grade 2 and all multiplication and division combinations by the end of grade 4. Even the NCTM *Standards* (1987)² and the new British Columbia program hold the traditional view that by the end of grade 4 all basic number combinations should be memorized - specifically, producing a correct response to a presented combination within 3 seconds.

A recent study (Hope, Baker & Mansfield, 1987) of the basic number combinations involving approximately 1750 students in grades 2 to 12 suggests that this traditional view is unrealistically optimistic because automaticity rarely develops by the end of grade 4, despite the considerable efforts of teachers. In this study, 25 difficult and 5 easy number combinations involving each operation were presented visually on a television monitor. Depending on the grade level of the sample being tested, students were given either 9, 6, or 3 seconds to write down an answer before a subsequent combination was presented.

Students were considered to have memorized the basic number combinations involving an operation when they produced correct responses to 90% or more of the items presented at a 3-second rate. Figures 1 to 4 show the percentages of students at each grade level who demonstrate mastery of the addition, subtraction, multiplication and division basic number combinations.

As can be seen from examining the figures, the majority of children memorize the number combinations several years after the grade level specified in most programs. The following information identifies the number operation and grade level at which the number combinations involving that operation can be said to have been memorized by the majority of children: addition, grade 4; subtraction, grade 5; multiplication, grade 6; division, grade 6. By the end of grade 4, where students are expected to have memorized most combinations, the percentage of students who could adequately recall the addition, subtraction and multiplication combinations were 55%, 20%, and 30%, respectively. The study also found that performance slipped

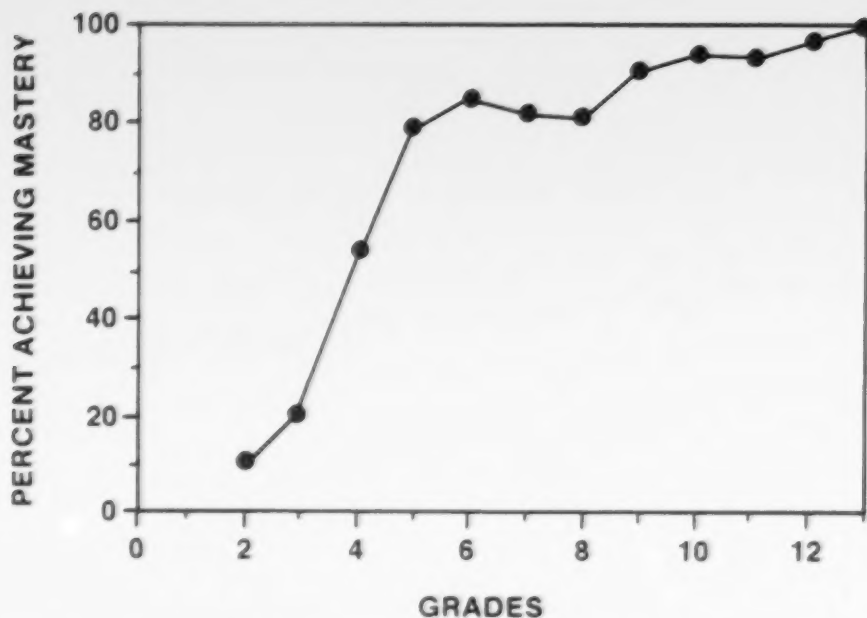


Figure 1. Percentage achieving mastery of the addition combinations presented at the 3-second rate (13 = Adult).

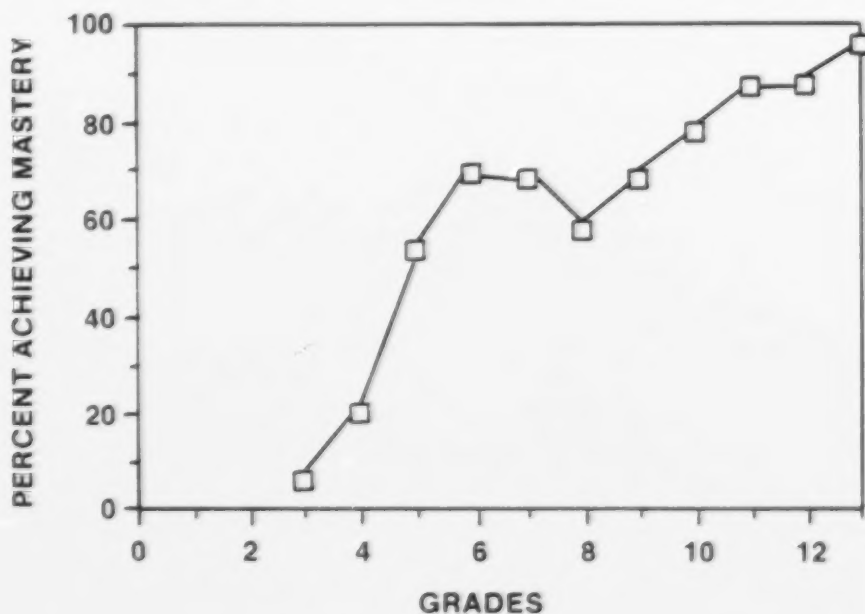


Figure 2. Percentage achieving mastery of the subtraction combinations presented at the 3-second rate (13 = Adult).

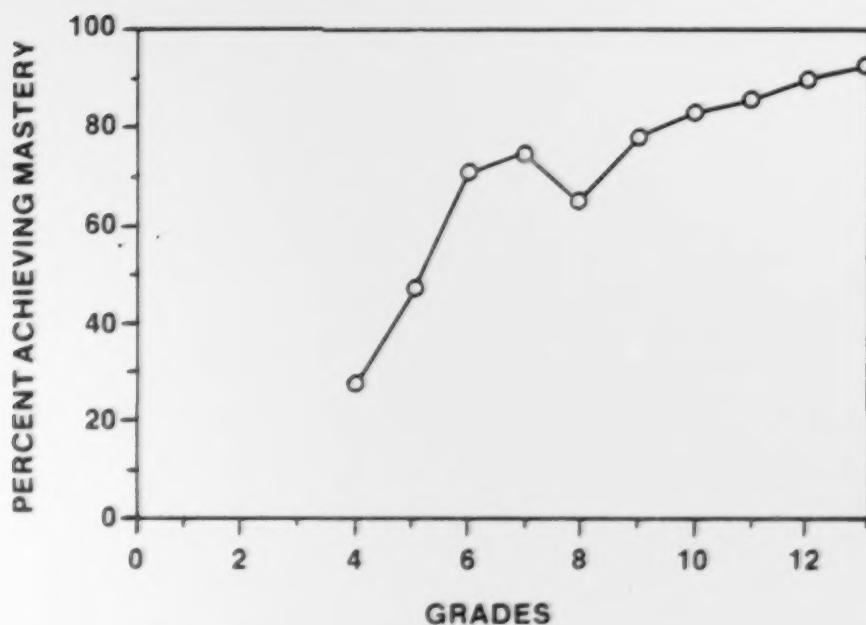


Figure 3: Percentage achieving mastery of the multiplication combinations presented at the 3-second rate (13 = Adult).

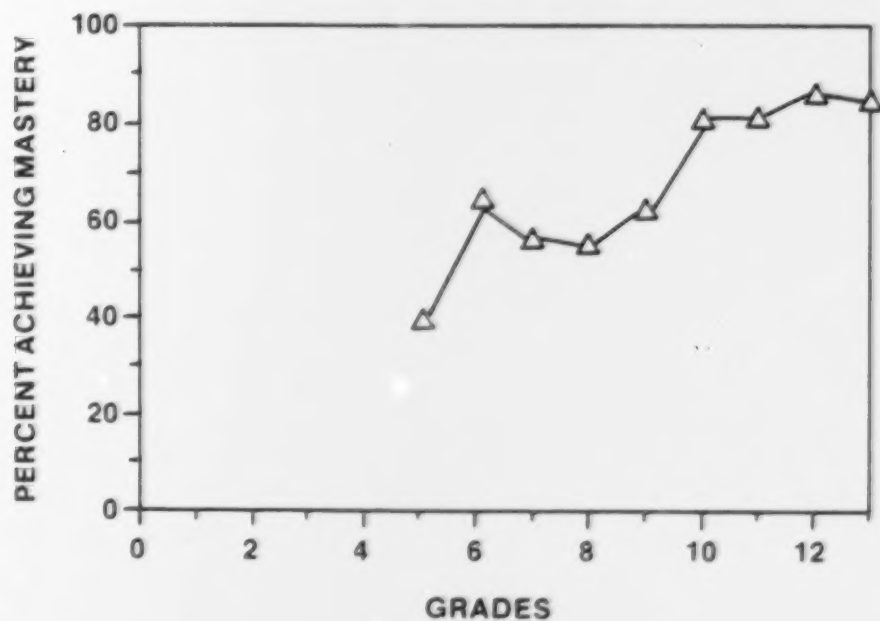


Figure 4: Percentage achieving mastery of the division combinations presented at the 3-second rate (13 = Adult).

in the later middle grades: for example, 68% of grade 6 students could adequately recall the subtraction combinations, compared to 56% of grade 8 students.

When given ample time to respond, however, the performance of elementary grade students is very high: for example, the median scores on the 9-second addition test administered to grade 2, 3, 4 and 5 students ranged from 97% to 100%. Similarly high scores were also evident on the other 9-second basic number combination tests. The substantial improvement in performance resulting from being given more time to respond suggests that most younger students are calculating rather than recalling the basic number combinations.

The results of this study suggest that automaticity is not attained until several years after a child leaves the primary grades. Consequently, it seems unreasonable to expect elementary teachers to be chiefly responsible for developing automatic recall of all basic number combinations. Elementary teachers should be principally responsible for ensuring that their students have memorized some of the simpler combinations and, more importantly, can use this knowledge to reason out any unmemorized basic number combinations as well as to solve simple problems involving recalled and reasoned out number combinations. Upper elementary and middle grade teachers need to share the responsibility for developing and maintaining automatic recall of the basic number combinations, especially the more difficult ones. Although facts cannot be categorized rigidly as either difficult or easy, Table 1 lists some of the facts identified in the Hope, Baker, and Mansfield study as being especially difficult.

Table 1. Five Most Difficult Combinations for Grades 2 to 6 Students to Produce a Correct Answer within 3-seconds

<u>Addition</u>	<u>Subtraction</u>	<u>Multiplication</u>	<u>Division</u>
6 + 8	13 - 7	3 x 8	27/3
5 + 8	14 - 9	7 x 9	24/3
6 + 7	11 - 8	7 x 8	63/9
3 + 8	11 - 7	8 x 6	54/6
8 + 7	15 - 6	4 x 9	18/6

Table 2 is a modification of a chart presented in the first draft of the Standards. In this chart, I have attempted to establish reasonable expectations for learning and eventually memorizing the basic number combinations.

Table 2. Reasonable Expectations for Learning the Basic Facts in the Elementary and Middle Grades

Operation	Grade					
	1	2	3	4	5	6-9
+ easy facts ⁵	e ¹ , f ² , s ³	s	m ⁴	m	m	u ⁷
+ hard facts ⁶	e, f, s	s	s	m	m	u
- easy facts	e, f, s	s	m	m	m	u
- hard facts	e, f, s	s	s	s	m	u
x easy facts	e	e	f, s	s	m	u
x hard facts	e	e	f, s	s	s	u
÷ easy facts	e	e	f, s	s	m	u
÷ hard facts	e	e	f, s	s	s	u

¹ explore refers to experiences that make use of objects to model facts that occur in the context of a problem. The language is informal (5 and 4 more makes how many; show how these counters can be shared among 3 children) and neither the symbols nor mathematical terminology is introduced.

² formalize refers to experiences used to help children learn the symbols for the operation and the associated mathematical terminology (6 divided by 2 equals 3, 4 plus 3 equals 7).

³ strategy refers to experiences that help children learn efficient reconstruction strategies to answer basic fact problems (4 fives equal 20, so 5 fives is one more five). Children are encouraged to explain their reasoning.

⁴ memorization refers to an expectation to memorize the facts. Practice is designed to help children respond eventually within 3 seconds after a basic fact stimulus is presented.

⁵ easy facts refers to those facts that children easily memorize (the addition doubles, adding 0, multiplying by 1) or where counting is easily applied ($6 - 1 = 5$, $8 + 2 = 10$).

⁶ hard facts refers to those facts that children find difficult to memorize ($8 + 7$, 7×8) and are usually reasoned out.

⁷ use activities to maintain recall; use practice focussing on only the unmemorized combinations.

Reconstruction Strategies

Research has demonstrated that recall can be greatly facilitated when children learn a number of different ways to reconstruct unknown combinations from those already memorized. Instead of organizing instruction around the rote memorization of "families" of number combinations (sums of 5, the 7 times table, etc.), as is presently done in textbooks, instruction in the combinations should be organized largely around reconstruction strategies (also referred to as "thinking" or "reasoning" strategies), which will be identified later in this chapter.

The use of a strategy by children depends entirely on how well the underlying principles and concepts are understood. The development of the necessary understanding can be accomplished by exposing children to carefully selected concrete referents, number patterns and visual patterns. Children should develop a good understanding of a strategy and be proficient in its use before other strategies are introduced.

Once children begin to recall rather than reason out previously unknown basic number combinations, teachers need to help children apply their knowledge of the reconstruction strategies to figure more difficult computations. For example, if children understand that $6 + 9$ can be reasoned by adding 10 and subtracting 1, they can learn eventually to apply this reasoning to the following more difficult calculations:

1. $29 + 56 = 56 + 30 - 1 = 86 - 1 = 85$
2. $199 + 256 = 200 + 256 - 1 = 456 - 1 = 455$
3. $\$9.99 + \$8.99 = \$10 + \$9 - 2 \text{ cents} = \$18.98$

Strategies used initially to reason out the basic number combinations should not be allowed to atrophy once children have learned the combinations by heart because they likely contribute significantly to the development of mathematical thinking.

Addition Strategies

Several reconstruction strategies to figure difficult addition facts can be identified: 1) counting on from the larger addend; 2) counting on or back from a known sum; 3) building on a known sum; 4) complements of 10.

1. Counting on from the larger addend. This strategy is best used when the addends differ greatly in size: e.g., $2 + 7$ can be reconstructed as $7 + 2$ and the sum figured by counting on by ones from 7 to 9. In some cases, larger counting units can speed up

the process such as counting in steps of 4 when adding 8: e.g., $8 + 8$ is counted as 12, 16.

To apply this strategy successfully, children need to be able to count on from a number, recognize the larger addend, and keep track of the count. To discourage children from counting all the objects rather than counting on when figuring a sum, it is recommended that the larger addend be represented with immediately recognizable quantities (e.g., adding pennies to a nickel instead of pennies to pennies). Number charts listing the multiples of the digits can be used to assist children in counting on from an addend in multiples greater than a single unit.

Children should be taught to extend counting on to carry out more complex tasks such as finding the sum of 2 cents and 47 cents or finding the length of 3 cm and 65 cm. Counting on is often continued into adult life to carry out simple tasks: e.g., adding $3\frac{1}{2}$ cups and $1\frac{1}{2}$ cups by counting on from $3\frac{1}{2}$ in increments of $\frac{1}{2}$ cup.

2. Counting on or back. As soon as children begin to recall certain sums, they should learn to recognize situations where unknown sums can be quickly reasoned by counting on or counting back from known sums. Concrete problems such as the following can promote the development of this strategy: "If we know a nickel and 2 more cents equals 7 cents, how can this fact help us in figuring out the sum of 5 cents and 3 more cents?"

3. Building on a known sum. Special emphasis should be placed on figuring sums by building onto either a known double or known sum of 10: knowing that 6 eggs and 6 eggs equals 12 or a dozen eggs can be used to figure sums such as $6 + 7$, $6 + 8$, $6 + 5$, for example.

The visual referents listed in Table 3 serve as a helpful cues to help children learn and apply the doubles combinations. If the double $10 + 10 = 20$ is known, larger sums such as $9 + 9$, $9 + 8$, $9 + 7$, $8 + 8$, and $8 + 7$, can be figured by counting back or subtracting from 20. Doubles can also be used to reason out the sum of "neighboring numbers" such as $5 + 7$, $6 + 8$, and $7 + 9$ by equalizing quantities: e.g., equalizing sets of 5 and 7 objects to produce two sets of 6 objects.

Table 3. Common Referents for Learning the Addition Doubles

<u>Double</u>	<u>Referents</u>
1 + 1	eyes, hands
2 + 2	car tires, dog legs, chair legs
3 + 3	insect legs, pop cases
4 + 4	spider legs, octopus tentacles, 8-carton pop case
5 + 5	fingers on both hands, toes on both feet, two nickels
6 + 6	egg carton, dozen compartment muffin tray
7 + 7	days in two calendar weeks
8 + 8	crayons in a standard carton
9 + 9	two baseball teams, tires on an 18-wheeler truck

4. Complements of 10. To encourage the learning of the complements of 10, orally presented problems such as the following are recommended: e.g., "How much change for a dime would you receive, if you buy a candy costing 3 cents?"

Subtraction Strategies

Several reconstruction strategies for subtraction can be identified: (1) counting back; (2) counting up; (3) using known facts; (4) using complements.

1. Counting back. Children commonly count back to determine differences, applying the procedure in two distinctive ways: (1) by counting back from the minuend the size of the subtrahend (9 - 2 is reasoned by counting back 2); (2) by counting back from the minuend to the subtrahend (9 - 7 is reasoned by counting back until 7 is reached). When the difference between two numbers is no greater than 3, both of these ways can be efficiently used to subtract.

The accuracy and speed of this strategy are greatly affected by both the size of the subtrahend and the size of the difference. Most children have difficulty reasoning out subtractions like 13 - 7 and 17 - 8 because several mental processes must be coordinated simultaneously: namely, counting backwards while keeping track of the progress of the count. (Adults can appreciate children's difficulty by trying to name the 7th letter preceding the letter r.) The efficiency of this strategy can be increased, however, by increasing the size of the counting steps: for example, subtracting 8 from 16 by counting backwards in steps of 2 or 4 rather than in steps of 1. Because of the excessive information-processing demands involved in subtracting by counting backwards in single units, children should be encouraged to use other more efficient subtraction strategies.

The activities previously presented in this paper to teach counting on can be adapted to teach counting back: e.g., counting backwards on a ruler to calculate the difference between 10 and 8. This strategy can be extended to carrying out tasks involving computations like $50 - 2$ or $135 - 133$.

2. Counting up. To eliminate some of the difficulties encountered by children when subtracting by counting backwards, researchers recommend that introductory instruction in subtraction emphasize counting forward procedures: i.e., counting up from the subtrahend. Tasks that involve comparisons are recommended: e.g., figuring the difference between collections of 7 and 9 objects by adding 2 objects to the collection of 7.

In carrying out practical tasks involving differences, most adults prefer to use counting forward rather than use a formal subtraction algorithm: e.g., finding the difference between 2 m and 1.98 m by counting forward from 1.98 m to 2 m in increments of either 1 cm or 0.01 m.

3. Using known facts. When children begin to recall rather than reason out various addition and subtraction combinations, unknown combinations should be reasoned by recasting them into sums and differences involving known combinations: i.e., figuring from a known sum or difference. The accuracy and speed in figuring subtraction can be facilitated by knowing the doubles combinations and recognizing complements of 10: e.g., $12 - 6 = 6$ because $6 + 6 = 12$ and $10 - 8 = 2$ because $8 + 2 = 10$.

As soon as children can recall some of the difficult addition combinations, they should learn how to apply this knowledge to reason out unknown subtraction combinations. Exercises such as the following that relate addition and subtraction facts are helpful:

Explain how the addition fact below can help in figuring the subtraction facts

$$6 + 5 = 11$$

$$11 - 5 =$$

$$12 - 5 =$$

$$11 - 6 =$$

$$12 - 5 =$$

Figuring from a known sum or known difference can be used to carry out a wide number of routine tasks: e.g., calculating the difference between \$10.37 and \$8.99 either by adding on to \$9 then to \$10.37 or by subtracting \$9 and then adding back 1 cent.

4. Complements. The complementary method of subtraction can be used to handle the difficult "teen" subtraction combinations as shown in the example below. The

strategy can be taught by representing 15 as 1 ten plus 5 units, removing 8 units from the 10, and counting the remainder ($5 + 2 = 7$).

$$\begin{array}{r} 15 \\ - 8 \\ \hline \end{array}$$

Subtracting 8 from 10 leaves 2

$$\begin{array}{r} 15 \\ - 8 \\ \hline 7 \end{array}$$

Adding 5 and 2 equals 7.

Multiplication Strategies

Several reconstruction strategies for multiplication can be identified: (1) skip counting; (2) distribution; (3) factoring; (4) doubling.

1. Skip counting. As soon as students realize that multiplication can be considered as repeated addition, they naturally reason out products using rapid skip counting: e.g., 7×5 is counted 5, 10, 15, and so on. This strategy is effective when either the size of the count or the number of counts is relatively small. To be proficient, children have to be able to skip count quickly and accurately while keeping track of the count. Children can also learn to abridge skip counting by looking for a pattern: e.g., 7×5 can be reasoned as 10, 20, 30 and another 5 equals 35. Whenever possible, skip counting should be practiced in the context of carrying out some practical task: e.g., figuring money amounts by counting out nickels.

Unfortunately, left to their own resources, some children never learn any other strategy to multiply than skip counting.³ Consequently, other less broadly applicable but more efficient strategies must be taught. But students should learn that skip counting is used in adult life to carry out a number of practical tasks such as counting out quarters (25 cents, 50 cents, 75 cents, etc.) or measuring out kitchen cups ($1/2$ cup, 1 cup, $1\frac{1}{2}$ cups, 2 cups, etc.).

2. Distribution. Additive and subtractive distribution are perhaps the most important arithmetic principles taught in elementary school. Children need to understand that unknown products can be determined by splitting one factor into several easily figured or recalled partial products: e.g., $6 \times 8 = 5 \times 8$ plus 1×8 and $6 \times 9 = 6 \times (10 - 1) = (6 \times 10) - 6$. These principles should be modelled using rectangles, arrays or other appropriate concrete materials.

To minimize confusion, children should be encouraged to translate a multiplication statement into a meaningful expression involving commonly seen groupings before carrying out distribution. For example, children find it easier to distribute 6×9 if they

think of expressions like "6 nines equals 5 nines and 1 more nine" or "6 teams of baseball players equals 5 teams and 1 more team."

Distribution underlies many types of mental calculation strategies that should be taught in the later elementary and middle years:

$$1. 8 \times 45 = 8 \times 40 + 8 \times 5 = 320 + 40 = 360$$

$$2. 8 \times 59 = 8 \times 60 - 8 \times 1 = 480 - 8 = 472$$

$$3. 6 \times \$1.25 = 6 \times \$1 + 6 \times 25 \text{ cents} = \$6 + \$1.50 = \$7.50$$

$$4. 4 @ \$2.99 = 4 @ \$3.00 \text{ less } 4 @ 1 \text{ cent} = \$12.00 - 4 \text{ cents} = \$11.96$$

$$5. 4 \text{ recipes of } 2 \frac{1}{2} \text{ cups} = 4 \text{ of } 2 \text{ cups and } 4 \text{ half cups} = 8 \text{ cups} + 2 \text{ cups} = 10 \text{ cups}$$

3. **Factoring.** Some unknown multiplication combinations can be reasoned by factoring one of the factors, recalling a known combination, and finally applying either a mental addition or mental multiplication strategy: e.g., $8 \times 7 = 2 \times (4 \times 7) = 28 + 28 = 56$. This strategy can be introduced to children as a way of rearranging a grouping of objects as illustrated below:

$$6 \times 5$$



$$3 \times 10 = 30$$

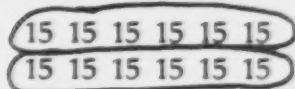
The strategy can be extended to more difficult calculations in the same manner:



$$6 \times 30 = 180$$



$$3 \times 60 = 180$$



$$2 \times 90 = 180$$

4. **Doubling.** Repeated doubling is a special form of factoring that becomes evident when the multiplication tables for 2, 4 and 8 are compared. A similar pattern is evident when the tables for 3 and 6 (and 12) and for 5 and 10 are compared. These patterns can be developed into a useful reconstruction strategy: "I know 4 fives equal 20, so 8 fives are double 20 or 40."

To carry out repeated doubling successfully, students must know how to double 2-digit numbers mentally: e.g., reasoning the difficult combination 6×9 as "double (3×9); double 27; double $20 = 40$, double $7 = 14$; 40 plus $14 = 54$." Regular oral drills involving practical situations such as the following are recommended: e.g., "Figure the cost of two pencils each priced at 28 cents".

Factoring and doubling can be used to carry out many types of seemingly difficult mental calculations:

$$1. 12 \times 12 = \text{double } (6 \times 12) = \text{double } 72 = 144$$

$$2. 18 \times 15 = 9 \times (2 \times 15) = 9 \times 30 = 270$$

$$3. 24 \times 25 = 6 \times (4 \times 25) = 6 \times 100 = 600$$

Division Strategies

The following strategies can be used to reason out the division combinations: (1) using multiplication; (2) distribution; (3) factoring.

1. Using multiplication. The Standards argue that since most division problems involve remainders, it is unnecessary and unwise to spend an excessive amount of time on memorizing division facts. Children and adults are far more likely to recast a division problem into one involving multiplication and recall known multiplication facts than to recall division facts. Thus, students need to master the multiplication combinations and learn how to use multiplication to solve division problems.

Before children begin a formal study of division, they should be able to use a number of multiplication reconstruction strategies and recall most multiplication combinations. Much of the instruction on learning the division combinations should focus on helping children realize that unknown division combinations can be figured by applying a suitable multiplication strategy or by recalling a known multiplication combination.

The number balance is an effective way to help children establish the connection between multiplication and division because multiplicative thinking is required to carry out division: e.g., to figure 15 divided by 3 requires that 5 weights be placed on location 3, which in turn represents the product of 5 and 3. Regular drills that emphasize the relationship between multiplication and division are recommended: e.g., "How many nickels are needed to equal 35 cents? State this problem using multiplication and then division."

Each multiplication strategy discussed earlier has a counterpart that can be used initially to reason out unknown division combinations and later extended to handle more difficult calculations. Consequently, only two additional reconstruction strategies for division will be discussed.

2. **Distribution.** In carrying out divisive distribution, the dividend is rearranged into a sum or difference of multiples of the divisor and then each term is successively divided:

$$455 \div 5 = (40 + 5) \div 5 = 8 + 1 = 9$$

$$81 \div 9 = (90 - 9) \div 9 = 10 - 1 = 9$$

This strategy can be extended to more difficult calculations:

$$515 \div 5 = (500 \div 5) + (15 \div 5) = 103.$$

3. **Factoring.** In carrying out divisive factoring, the dividend is successively divided by factors of the divisor:

$$24 \div 4 \quad 1/2 \text{ of } 24 = 12, \quad 1/2 \text{ of } 12 = 6$$

$$48 \div 6 \quad 1/2 \text{ of } 48 = 24, \quad 1/3 \text{ of } 24 = 8$$

This strategy can be extended to more difficult calculations:

$$144 \div 6 \quad 1/2 \text{ of } 144 = 72, \quad 1/3 \text{ of } 72 = 24.$$

Promoting and Maintaining Recall

Carl Sandburg's epigram seems an appropriate way to introduce a discussion of drill and practice:

Arithmetic is where you have to multiply - and you can carry the multiplication table in your head and hope you won't lose it.

To carry out routine calculations and straightforward practical tasks efficiently, children eventually need to be able to recall rather than reason out most of the basic number combinations. An essential ingredient in helping children master the basic number combinations, especially those that are difficult for children to reconstruct, is to provide practice where it is clearly understood by the children that the intent is memorization.

Because practice increases the speed and accuracy of a response but not the quality of thinking, it should be used sparingly and never as a substitute for teaching children to reason out the combinations. Practice can be effective only when children receive **instant feedback** regarding the accuracy of a response. If this feedback is absent or delayed, the practice may serve to habituate incorrect rather than correct responses; unmonitored practice is probably one of the major reasons that incorrect associations persist for years, despite the considerable practice provided by teachers. Consequently, the practice sets of exercises commonly seen in workbooks and textbooks and on teacher-constructed worksheets are usually **ineffective** because feedback is normally absent or too delayed to be useful. Flashcards, number wheels, calculators and pre-programmed calculators such as Dataman and Little Professor, pre-recorded audiotapes and records, and computer programs are a few of the many inexpensive devices designed to provide children with instant feedback about the adequacy of their responses.⁴

For those children who are accurate but respond slowly, the practice activities should be designed to **limit the amount of time available to respond** to a presented number combination. When children have unlimited time to respond, they often continue to use less efficient counting procedures to reason out combinations: counting fingers is a commonly seen behaviour. By limiting the time to respond, the inefficiency of the method becomes more apparent and children are provided with an incentive to learn more efficient reconstruction procedures or to memorize the unknown combinations.

A practical way to limit the time available for a response is to record and present the combinations orally on audio tape and to fix the time available for recording a written response. A portion of a tape presentation designed to practice the addition combinations is described below:

"Question one: Four plus three equals?" (time interval 1)

"Answer one: Four plus three equals seven." (time interval 2)

"Question two: Eight plus seven equals?" (time interval 1)

"Answer two: Eight plus seven equals fifteen." (time interval 2)

An initial item presentation rate of about 6 or 7 seconds (time interval 1) encourages a rapid response without frustrating slower responding children. Children should be given about 3 or 4 seconds (time interval 2) to correct an incorrect response and to rehearse the correct combination (write it or say it silently) before another item is presented. Those children encountering difficulties in memorizing should be encouraged to use these tapes at home or at a private listening post in the classroom.

When children become very proficient with a practice tape, the time allowed for a response (time interval 1) can be gradually decreased. For all practical purposes, a combination can be assumed to be memorized if an accurate response is produced within 3 seconds after the combination has been presented. However, teachers are in the best position to determine whether this response time is a realistic one for the children in their classrooms.

Practice should centre on **memorizing the difficult rather than the easy combinations**. According to Hamann and Ashcraft (1986), textbooks do not provide enough practice sets involving the more difficult number combinations. Their analysis of widely used primary grade textbooks reveals that the easily mastered addition combinations involving addends of 1, 2, 3 and 4 are presented much more frequently than the difficult combinations involving addends of 6, 7, 8 and 9. Although the researchers do not suggest that frequency of textbook presentation is the sole, or necessarily the most important, factor in children's memorization of the addition combinations, they believe that it must be considered an important one. Consequently, teachers need to ensure that their practice activities concentrate on the combinations that children find difficult to remember rather than rely entirely on those combinations presented in textbook and workbook practice sets.

The **practice can be personalized** by using the results of teacher-prepared tests. Each child can be instructed to select and study daily a few combinations taken from a list of the number combinations that the test results indicate are personally difficult. Children gain personal satisfaction when they realize after subsequent re-testing that their list of difficult combinations gradually contains less and less items.

To ensure that children know how to apply their knowledge of number combinations to carry out simple tasks and to solve straightforward problems, **regular practice with orally presented situations** such as the following is highly recommended:

1. A nickel and 3 pennies equals how much money?
2. Seven nickels equals how many pennies?
3. How many bottles are in four cases of pop?
4. How many cars can be outfitted with 36 tires?

The more closely a recall task resembles the representational form in which information is stored, the more likely the information can be retrieved successfully. For example, if practice is restricted to representational forms requiring the recall of a product (e.g., $6 \times 5 = ?$), children find that retrieving information presented in a

different representational form (e.g., $6 \times ? = 30$, $5 \times ? = 30$) is a considerably more difficult task because the memorized information is stored in different form.

Children who can recall the multiplication combinations cannot necessarily readily carry out more complex tasks calling for the retrieval of a number combination. For example, finding the greatest common factor of 48 and 64 requires more than the straightforward retrieval of a product; instead, children must reverse their thinking and recall factors rather than a single product. Because number combinations are seen and used in differing forms, children should be encouraged to **memorize these different representation forms**. Much of this type of thinking should be delayed until the middle years.

Similarly, children need to learn to associate the various written forms for representing the number combinations with their equivalent oral forms. For example, all four oral expressions listed below are equivalent to the written form $6 \times 9 = ?$: (1) "What number is six times nine?" (2) "What is the product of six and nine?" (3) What is six multiplied by nine?" (4) "How much is six nines?" Teachers should not restrict the practice activities to the written forms seen in textbooks and workbooks. Instead, **regular practice with the oral forms** should also be provided.⁵

To minimize boredom, the practice activities should be changed regularly to make them at least palatable, if not enjoyable. **Game-like activities** are especially recommended. Games involving the number combinations can be purchased from most school suppliers. Although many commercially-prepared games are available, useful practice activities can be created from most popular dice games, card games, T.V. game shows or sports activities. For example, the card game Blackjack can be used to provide children with practice in adding to a sum of 21.

Children who are struggling with the basic number combinations find practice activities based on games of chance enjoyable since they require less knowledge than games based entirely on skill. Games provide children with an incentive to become more accurate and to respond faster. The other competitors can be relied upon to provide the necessary feedback, especially when a competitor's response is incorrect.

Of course, teachers need to be cautious about the competitive nature of games; some children become upset when they lose. Teachers should use games not only to help children practice number ideas, but to learn how to accept losing graciously. Games that have an element of chance such as dice-and-board games ensure that even the least able children have a chance to succeed.

Footnotes

- ¹ The basic number combinations, sometimes referred to as basic facts, are defined as any combination involving an operation and two single digit numbers. Combinations like $10 + 10 = 20$ are not considered basic number combinations, although they are easily memorized. Ignoring the commutative property of $\times$ and $+$, the number of combinations for each operation are addition (100), subtraction (100), multiplication (100) and division (90).
- ² The 1987 draft report of the Standards was unequivocal in its expectations: recall of all basic combination by the end of grade 4 (p. 38). The 1989 final report is very ambiguous: "Children should master the basic facts of arithmetic that are essential components of fluency with paper-and-pencil and mental computation and with estimation. At the same time, however, mastery should not be expected too soon" (p. 47).
- ³ I once worked with a 15-year-old who when presented with a multi-digit multiplication computation (e.g., 35×487) would immediately proceed to draw tally sticks to represent each single-digit partial product and determine these products by skip counting. Needless to say, he rarely finished a computation quickly and correctly.
- ⁴ Fuson and Brinks (1985) designed a flash card procedure with some of the features of microcomputers for the drill and practice of basic facts. Children practised basic facts in either a flash card or microcomputer condition. The flash card and microcomputer conditions produced equivalent learning.
- ⁵ Children who are able to recall multiplication facts given on a written test are often confused by parent's oral queries (for example, "What's 6 times 5?"), leading to the parent's erroneous conclusion that schools are not teaching the "times tables" anymore.

CHAPTER 9

COMPUTATIONAL KNOWLEDGE

As has been argued in Part 1, the current standards for computational proficiency are unrealistic and need to be revised to take into account the advent of the electronic, portable calculator. Studies have shown that most teachers, however, are spending an excessive amount of time teaching outdated skills. For example, the Cockcroft report (1983) cited a British survey that found in about a third of primary classrooms "children were spending too much time undertaking repetitive practice of processes which they had already mastered." Furthermore, "the efforts made to teach children, to calculate are not rewarded by high scores in ... examples concerned with the handling of everyday situations. Learning to operate with numbers may need to be more closely linked with learning to use them in a variety of situations than is now common" (p. 88).

A survey of British first schools (5 to 9 years) discovered that in only one fifth of the schools was there "a good balance between learning how to perform a calculation and using it in a practical setting" (Hughes, 1988, p. 9). In the remaining classrooms, skills were practised in isolation from practical problems. The survey further noted that "written calculations often formed the main element in mathematics work, and in some schools the sole aim seemed to be for the children to reach a standard of efficiency in abstract calculations" (p. 9).

Similar findings were reported in a study of British junior schools (8 to 11 years). About 82% of the teachers surveyed reported that they placed foremost emphasis on basic computational skills - learning multiplication tables and number facts. One researcher concluded that there is no need to exhort the vast majority of teachers to "go back to the basics" as "the development of and practice in the basic skills of English and mathematics would seem to be the predominant features of junior school classrooms" (Hughes, p. 9).

In the 1985 British Columbia Mathematics Assessment (Robitaille & O'Shea), teachers were asked, "What percent of the last mathematics period you taught did your students spend on computational drill?" Only about 25% of the grade 4 teachers and about half of the grade 7 teachers reported that no time was spent on drill. In comparison, over 80% of grade 4 teachers and about 90% of grade 7 teachers reported that students spent no time working at stations or activity centres. Clearly, as these studies demonstrate, unless teachers reduce their efforts to teach traditional computational skills, the more important activities involving problem solving and critical

thinking will continue to be neglected and the mathematics program of tomorrow will continue to resemble those of the past.

In a contemporary elementary and middle years mathematics program, most children who complete the program should be expected to have acquired the following competencies:

1. Know how to carry out **straightforward** calculations using either pencil-and-paper or mental methods of computation.
2. Understand the number principles underlying a calculation method.
3. Know how to operate a calculator to carry out appropriate calculations.
4. Use a calculation method to solve a problem or to carry out a task and interpret the meaning of a calculated answer using the context of the problem or task.
5. Know a variety of ways to determine the reasonableness of a calculated answer.

Pencil-and-Paper Calculation

Without question, children still need to learn some traditional methods of written calculation. However, as a general rule, students should be taught **only those written routines commonly used in daily life using reasonably sized numbers**.

Standards of Performance

By the end of year 5, the last year of the elementary program, most children should be able to perform the following types of written calculations:

1. Adding several 2- to 4-digit whole numbers.
2. Adding several decimal numbers each expressed in up to 2 decimal places.
3. Subtracting two 2- to 4-digit whole numbers.
4. Subtracting two decimal numbers each expressed in up to 2 decimal places.
5. Multiplying a 1-digit whole number by a 2- to 4-digit number.
6. Multiplying a 1-digit whole number by a decimal number expressed in up to 2 decimal places.
7. Dividing a 2- to 4-digit number by a 1-digit whole number.

8. Dividing a decimal number expressed in up to 2 decimal places by a 1-digit whole number.

The written computational standards for the middle years should be expanded to include some "limited" work with 2 digit factors and divisors. Students should also be expected to add, subtract, multiply and divide commonly seen fractions. The more difficult computational work involving fractions such as division should be delayed until year 8 or 9, the last years of the middle grade program.¹ Students should learn informal as well as the standard, formal methods of computation: for example, calculating a square root using a guess-and-check method. Integer arithmetic should also be included in the later middle grades.

The Development of Understanding

Paper-and-pencil algorithms need be taught in a way that **emphasises understanding of the number principles underlying these algorithms** - i.e., knowing why the algorithms work. To acquire this understanding, children will need to work with appropriate concrete materials, usually the materials used to develop place value and fractional concepts. The task for teachers is to help children connect the actions involved in calculating with these materials and each step in the associated algorithm. This provides teachers with a unique opportunity to involve children in the processes of problem solving: discussing, hypothesising, validating, and generalizing.

Once children have acquired an understanding of and a facility with an algorithm, the learning of new algorithms can be based on this knowledge rather than on work with concrete materials. For example, children who know how to multiply a 1-digit whole number by a 3-digit whole number with understanding find it relatively easy to predict how a 1-digit whole number can be multiplied by a 4-digit number.

Even if computation work is decreased, students will have to be provided with computational practice. But it is desirable that they practise computation while learning to solve problems rather than by working on computations stripped of a context. Textbooks that use a problem-solving approach to practise computation should be preferred to the traditional texts and workbooks that present exercises stripped of a context.

Teachers have to become better diagnosticians than they have been in the past.² Calculation is a difficult task for learning disabled students who often have difficulty remembering each step and its order in an algorithm. These types of students need instruction centering on their specific difficulties. For example, children who have difficulty in remembering to record a 'carry' number during addition should be provided with visual cues and exercises directly centering on this difficulty rather than being retaught the entire algorithm again and again.

The processes of renaming and exchanging are central to a good understanding of whole and decimal number computation. Students in both the elementary and middle years should learn these processes through the use of concrete manipulatives such as base 10 blocks, bean sticks, Cuisenaire rods, money, measurement materials, place value mats and charts, and various abacuses. These processes can be developed through good problem solving activities such as the following:

Students are divided into groups and each group is given a different place value device. Each group represents the number 135 using the materials. The object is to find a way to reach 500 by adding additional materials.

Students are divided into groups and each group is given a different place value device. Each group is asked to find a way to share 100 among their members: for example, how can "a flat of 100 be shared in four ways?

Teachers should avoid using the archaic and confusing language commonly used to describe computational steps: carry, borrow, bring down, "guzinta," move over, and so forth.³ Instead, the terms "regrouping" or "exchanging" should be used whenever possible.

Middle years teachers should ensure that students learn to compute with fractions and decimals through well planned concrete experiences. Students should work with paper folding, Cuisenaire rods, and so forth to determine why the peculiar algorithms involving fractions "work." Fractional computation has been traditionally taught by rote, in a "shut up and do this" manner. Consequently, very few adults can provide a reasonable explanation of why we divide fractions by "inverting and multiplying," for example.

In developing the meaning of the algorithms, analogies and number patterns can often be employed. For example, the following analogy provides a reasonable explanation of the addition of fractions:

To add 3 nickels and 2 dimes, we have to consider a common unit, which in this case, can be pennies or dimes.

3 nickels + 2 dimes = 3 nickels + 4 nickels = 7 nickels.

To compute the sum of $\frac{3}{4}$ and $\frac{2}{3}$, think 3 quarters and 2 thirds. We have to consider a common unit (common denominator) for quarters and thirds, which can be 12ths, 24ths, 36ths, so forth.

$$3 \text{ fourths} + 2 \text{ thirds} = 9 \text{ twelfths} + 8 \text{ twelfths} = 17 \text{ twelfths.}$$

This particular analogy can lead into the development of an alternative algorithm for division. The examples below demonstrate that numbers can be easily divided if we find a common unit:

$$60/20 = 6 \text{ tens} / 2 \text{ tens} = 3$$

$$800/200 = 8 \text{ hundreds} / 2 \text{ hundreds} = 4$$

$$600/20 = 6 \text{ hundreds} / 2 \text{ tens} = 60 \text{ tens} / 2 \text{ tens} = 30$$

Consequently, we can divide two fractions by considering a common unit:

$$5/6 \div 2/9 =$$

$$5 \text{ sixths} / 2 \text{ ninths} = 15 \text{ eighteenths} / 4 \text{ eighteenths} = 15/4$$

$$5/6 \div 2/9 = 15/18 \div 4/18 = 15/4$$

By spending less time on the development of computational performance, teachers can be provided with more time to ensure that students learn the underlying principles needed to understand written computation.

Electronic Calculators

These devices must be considered an integral part of learning to compute. Before the Hindu-Arabic numeration systems and the associated algorithms were introduced to medieval Europe, calculations were performed exclusively on an abacus. Consequently, history demonstrates that calculation methods are usually obsolesced by emerging technologies and, in the case of the **portable electronic calculator**, it **has replaced traditional pencil-and-paper algorithms as the primary tool of calculation**. Thus, we have a responsibility to ensure that all students in the elementary and middle years become proficient in using a calculator.

By the end of the elementary years, all students should be able to perform the following types of calculations on a conventional 4-function calculator:

1. Adding, subtracting, multiplying and dividing.
2. Using the additive constant function to solve problems involving skip counting.
3. Using a variety of approaches to correct keystroking errors: e.g., using the CE feature, compensating, etc.
4. Interpreting the displayed answer: e.g., considers the context of the calculation to determine the appropriate number of digits displayed in the answer.

5. Uses the following special function keys on a calculator: decimal, clear display, correct error, clear memory, recall memory.

Students in the middle years, should expand their knowledge of the 4-function calculator: e.g., working with additional functions such as the % and square root key. They should also learn to reconstruct and interpret calculations in a variety of ways: for example, interpreting the display of 0.666666 as $\frac{2}{3}$, 0.67, 67%, and so forth. They should also have an opportunity to work with scientific calculators that require an understanding of exponents. Except for their use in some games, calculators will rarely need to be used in the early elementary grades.⁴ Until children are able to understand the common mathematical operations and can enter, read, and interpret larger numbers, a calculator is of limited use.

In the 1985 B.C. Mathematics Assessment, students were asked several questions related to their knowledge of calculators. Less than half the grade 4 students selected the answer 6327 when presented with the question "The three digits in the display have been entered [a picture of a calculator was shown displaying 632]. Now the 7 key is pressed. What number will appear in the display?" About 15% of the grade 7 students answered the following question correctly: "The following sequence of keystrokes is carried out on the calculator 6 3 + 7 CE 6 = . What will likely be shown on the display." It appears from these results that students are not being provided with instruction on how to use calculators in the most efficient way.

To use a calculator efficiently requires a good understanding of mathematics. Students should learn a variety of ways to correct errors, check the accuracy of an answer, check the reasonableness of answers, and to speed up the calculation process. They should discuss what can be done if an incorrect number is entered: for example, entering 1.25 instead of 1.35 in determining a sum. They should have a good knowledge of number principles and relationships to check the accuracy of an answer: for example, after multiplying 4.578 and 3.5, dividing the answer by either factor to ensure that the other factor is presented in the display. They should have sufficient knowledge to detect unreasonable answers: for example, realizing that a calculated sales tax of \$15 on a purchase of \$28 is an unreasonably large amount and a key entry error must have been made. They should also learn how to abridge their calculations through the use of mental calculation or the use of a calculator memory.

Calculators should not be used to calculate with numbers that can be done mentally or manually: e.g., the basic number combinations, working with multiples of 10, and so forth. Children will have to **learn when it is and when it is not appropriate to turn to a calculator**. Nor should they be used to check manual or mental calculations. This practice not only seems fraudulent (why bother calculating in the first instance if a calculator is readily available?) but intimates that the calculator, not the user, is the ultimate authority in determining the accuracy or reasonableness of a calculation.

Although teachers need to exercise their common sense when using calculators in a classroom, students will learn to use calculators intelligently only through their regular use in solving problems, not by being denied access to them.

Saskatchewan Education can expect some resistance from teachers and boards to the use of calculators in school. In the B.C. Mathematics Assessment, over 80% of grade 4 teachers and about 35% of grade 7 teachers do not allow calculators in their class. And in many of the classrooms where calculators were permitted, they were used by students to check their computational work. Over 90% of the teachers, about half of the grade 4 students, and over 70% of the grade 7 students said they owned calculators.

The general attitude of the teachers in British Columbia towards calculator use in school seriously conflict with the views expressed in the recently completed British Columbia Royal Commission on Education (1988): "The use of calculators, except for simple computations, should become routine and all students should be provided with calculators as they are now provided with textbooks" (p. 98).

Mental Calculation

In a revised elementary and middle years program, children should **learn a number of different techniques for calculating mentally**.⁵ Most recently revised curricula in Canada and the U.S.A. and commercial textbooks have included mental calculation as an important form of mathematical learning. Several reasons for this renewed attention can be identified (Hope, 1986; Reys, 1984).

First, as argued in Chapter 1, **mental calculation is a practical life skill**. Despite the proliferation of calculators, it will remain a convenient calculation tool for solving a narrow range of quantitative problems.

Second, **mental calculation can improve the efficiency of pencil-and-paper calculation** and work with calculators as well. Proficient mental calculators often reduce the number of steps needed to work out a written calculation by performing some steps mentally: for example, in adding 25 and 26, a child might eliminate steps by reasoning, "I know 25 and 25 equals 50, so I'll write down 51 as my answer."

Third, because estimation is considered a desirable goal of instruction, the close relationship between estimating and calculating mentally implies that teachers cannot ignore instruction on mental calculation. For example, in estimating the product of 9 and 789, the resulting multiplication of 9 by 800 should be done mentally. **Mental calculation provides the cornerstone for all estimation procedures**, offering a variety of alternative algorithms and non-standard techniques for finding answers.

Fourth, there is enough evidence for us to believe that **mental calculation can lead to a better understanding of place value, mathematical operations, and number relationships**. Children often do written calculations mechanically, without a great deal of thought, simply applying the written procedures and divorcing themselves from any sense of working with quantities.⁶ Think about the low-level of thinking needed to multiply 5 and 275, for example:

$\begin{array}{r} 275 \\ \times 5 \\ \hline \end{array}$	"5 times 5 equals 25, put down the 5 and carry the 2. 5 times 7 equals 35, plus 2 equals 37. Put down the 7 and carry the 3. 5 times 2 equals 10, plus 3 equals 13. Put down the 13. The answer is 1375.
--	--

Clearly, such routine procedures as 'borrowing,' 'carrying,' 'moving over,' 'bringing down,' and so forth are so divorced from the quantities involved that it makes no difference whether we are dealing with units, tens or hundreds. We must remember that the generalizability and efficiency of the traditional written calculation procedures taught in school requires that the user ignore any reference to the quantities represented in the calculation. This careful attention to digits and bookkeeping rules rather than to numbers and number relationships can contribute to a fragmentary view of number relationships.

Several years ago, I observed such mechanical behaviour in my own daughter. As a year two student, she once brought home from school a practice sheet of subtraction exercises. I observed her using the traditional subtraction algorithm to calculate $13 - 9$ as indicated below:

$$\begin{array}{r} 0 \\ \cancel{1}3 \\ - 9 \\ \hline 04 \end{array}$$

Later I asked her, "Kim, what is thirteen take away nine?" She immediately replied, "Four." When I asked her to explain why she used a pencil-and-paper method to calculate $13 - 9$ when she knew from memory the correct answer, she replied, "Dad, that's what we have to do in school."

While working with a curriculum revision committee of elementary school teachers in Saskatoon I related this story about my daughter's mechanical behaviour. At the next meeting of our committee, one teacher told me she witnessed similarly inflexible behaviour among the majority of her year five students when they responded to the

computational exercise $100 - 99$ by carrying out the usual regrouping procedures involved in the subtraction algorithm instead of realizing immediately that the difference equals 1.

There is substantial evidence suggesting that children often conclude that looking for shortcuts or doing anything but calculating with pencil and paper is somehow "wrong." In a study (Baroody, Ginsburg & Waxman, 1983) that examined children's use of mathematical structure to solve problems, children were encouraged by the investigators to use any method they wanted to solve a problem, including using information from previously solved problems. Despite this encouragement, some primary children seemed to believe that using known information about number relationships to abbreviate their computational efforts was somehow "naught." For example, one child who used the previously calculated fact $7 + 5 = 12$ to reason $7 + 7 = 12$ commented, "I cheated on that one." The researchers reported that this type of attitude persisted despite their efforts to counter it.

Unlike written calculation, **mental calculation forces students to think about numbers and number relationships.** As students learn to manipulate numbers in their heads, they develop a keen number sense and experience increased confidence in their mathematical abilities. Such confidence ensures that these students will not have to turn to a machine or pencil and paper for every straightforward calculation encountered in daily life.

Despite its many attractive benefits, mental calculation has not played a prominent role in most contemporary mathematics programs and, possibly because of this neglect, children often have difficulties in carrying out straightforward mental calculations and seem enslaved to the written procedures taught in school. In a recent unpublished study done in classrooms in Saskatchewan and in Texas (Hope, Reys, & Reys, 1990), the mental calculation performance of second and fifth graders was assessed. (Grade 7 students were also in the study, but the results have not been analyzed.) Students were presented with either a straight calculation or problem situation requiring a calculation and given about 6 to 10 seconds to answer, depending on the item and grade level. Items were presented visually and orally. Students were directed to do "all problems in their head and to write down only the answer."

Second graders found it difficult to compute mentally the sums and differences of two digit numbers. Some sample items and the percent answering correctly follow: 83 minus 63 (6%), 75 take away 45 (9%), 49 minus 24 (1%), Double 24 (16%). The fifth graders experienced similar difficulties as indicated by the following samples of performance: 47 plus 29 (35%), 357 plus 299 (38%), $68 + 32$ (53%), $859 - 159$ (62%), 60 multiplied by 70 (33%), 300×40 (30%), $25 \times 14 \times 4$ (1%), 3800 divided by 10 (12%), 3500 divided by 35 (16%). Kevin delivers 38 newspapers each day. How many newspapers does he deliver in five days? (28%).

The language of the item had an impact on performance. For example, performance was consistently higher on subtraction items using the phrase "take away" than on similar items using the term "minus." The statement "What is 40 more than 50?" produced much lower performance (12%) than the equivalent statement "50 plus 40", which was answered correctly by 57 percent.

A sub-sample of grade 5 students were also asked to identify the method they would use to carry out a given calculation. The results reveal a strong preference for written methods even when in some cases we would expect that mental methods should be used: for example, less than half of the students said they would figure the calculation 945×1000 in their head; the remainder of the students were about evenly split in preferring to use either paper and pencil or a calculator.

In contrast to the difficulties of some children in calculating mentally, there are some children who without any formal instruction learn to handle even very difficult mental calculations. The following questions were presented orally to a 13-year-old girl participating in a study (Hope, 1987a) that investigated student's abilities to reason with numbers, especially in carrying out multiplication mentally without benefit of paper and pencil.

Question: "What is 87 times 23"?

Response: "2001. I thought 30 times 69 equals 2070, and I subtracted 69. The answer is 2001."

Strategy: $87 \times 23 = 3 \times 29 \times 23 = 29 \times 69 = (30 - 1) \times 69 = 30 \times 69 - 69 = 2070 - 69 = 2001.$

Question: "Try 123 times 456."

Response: "Is the answer 56 088? I first thought that 456×123 is $152 \times 41 \times 9$. Then I thought $19 \times 41 \times 8 \times 9$. I did 779×8 by thinking 5600, 560, and so 6160. 6160 and 72 is 6232. Finally I multiplied 6232 by 9 by thinking 9×62 is 558, which I know. So 55 800 and 9×32 . 55 800 and 288 is 56 088."

Strategy: $123 \times 456 = 41 \times 3 \times 3 \times 152 = 152 \times 41 \times 9 = 19 \times 8 \times 41 \times 9 = 779 \times 8 \times 9 = (700 + 70 + 9) \times 8 \times 9 = (5600 + 560 + 72) \times 9 = 6232 \times 9 = (6200 + 32) \times 9 = 55\,800 + 288 = 56\,088.$

Her parents informed me that she liked to do calculations mentally from the age of about 8 or so and they believed by the time I interviewed her that she had lost a great deal of her unique talent.

Guidelines for Instruction and Program Development

What can be done in the elementary grades to help children learn to calculate mentally? Obviously, the first step is to **identify the types of strategies that elementary school children are capable of learning**. However, a comprehensive list of teachable strategies is not available because of a lack of research in teaching mental calculation to children. I have developed a tentative list of strategies that appears at the end of this chapter. The reader should note that this list is not meant to be exhaustive. Some strategies are also listed in the section on basic number combinations, which is obviously another form of mental calculation.

Once strategies have been identified for each grade level in the elementary program, teachers need to be provided with guidelines to help them develop appropriate instruction. Four instructional steps can be identified (Hope, 1987).

Conceptualizing. In this step, the number principles underlying an emerging strategy are stressed. Concrete objects, number pattern and analogies are suitable starting points when trying to establish a conceptual base for a strategy. For example, working with money can provide the background knowledge for learning the mental multiplication strategy, aliquot parts:

- 1 dime = 2 nickels
- 2 dimes = 4 nickels
- 3 dimes = 6 nickels
- 4 dimes = 8 nickels

Continuing with this pattern, a generalization can be made: 16×5 can be reasoned as 16 nickels or 8 dimes or 80 cents

Formalizing. Once students have developed an intuitive understanding of a strategy a more efficient abridgement can be introduced and practised. Print materials such as Mental Math in the Primary Grades, Mental Math in the Middle Grades, and Mental Math in Junior High published by Dale Seymour can be used to help provide meaningful instruction and practice at this stage.

Practising. Eventually practice is needed to increase the speed and accuracy in using a mental calculation strategy. The children should also be encouraged to suggest alternative ways to calculate each practice exercise and to share their explanations with other children. Whenever possible, the practice exercises should involve real life problems such as the following:

1. How many buns in 5 dozen?

2. How much change from \$10 would you receive for a \$2.95 purchase?
3. How far will an auto travelling 90 km/h travel in 4 1/2 hours?

Children should be **constantly encouraged to use mental calculation to simplify calculations**. Exercises such as the one below are useful:

Use the completed calculation to figure the incomplete calculations:

- | | |
|----------------------|----------------------|
| $25 \times 25 = 625$ | a) $25 \times 26 =$ |
| | b) $25 \times 50 =$ |
| | c) $25 \times 2.5 =$ |
| | d) $625/25 =$ |
| | e) $25 \times 24 =$ |
| | f) $626/25 =$ |

Extending. The final step involves extending students' understanding of strategies involving smaller numbers to larger or different types of numbers. For example, if a child can use factoring to reason out a basic number combination (e.g., $6 \times 9 =$ doubling 27), this reasoning can be extended to more difficult calculations (e.g., $16 \times 25 = 4$ times $(4 \times 25) = 4$ times 100).

Additional Considerations

Because teachers view mental calculation as a "new" topic, they will have to be provided with instructional ideas such as the following:

1. Children should be encouraged to calculate mentally whenever possible without being forced to use methods they don't understand. We don't want mental calculation to become another form of rote learning.
2. Children should be expected to explain their methods of calculation and encouraged to use a variety of methods of solution. Student discussion contributes significantly to the learning of mathematics.
3. Whenever possible, mental calculation as with written calculation should be practised in a meaningful setting rather than as calculations stripped of a context. Examples of real-life problems in which mental calculation might be used should be presented regularly: for example, simulated purchases and change-making using advertising flyers and money.
4. Mental calculation should be treated as a skill to be used in all areas of the program rather than as a single topic studied once in a school year. Children need to be encouraged to use mental calculation in not only the mathematics period but in all

class periods, especially in science, social studies, health and physical education (e.g., how many teams of 4 can be made from our class of 26 students?).

5. Children, especially the more able students, should not be discouraged from skipping steps or using shortcuts when doing written calculations. They shouldn't have to repeat a calculation simply because they've not included all steps. Many able mental calculators lose their abilities to figure in the head when they are forced by insensitive teachers to write out every step of a calculation.

The creative teacher can always find ways to provide practice with written calculations yet not discourage the students from using mental calculation. For example, instead of having children complete an entire exercise sheet of two-digit addition, about half of the exercise could contain "compatible numbers" such as $75 + 25$, $81 + 19$, and so forth, and the children would be instructed to do **only** the calculations not equalling 100.

6. Regular practice and review in learning the basic number facts needs to be done in all upper elementary and middle grades. No mental calculation can be done without a good knowledge of these important number relationships. Also frequent practice in learning common numerical relationships should be given: for example, one dozen = 12, 60 minutes = 1 hour, 7 days = 1 week, 100 cm = 1 metre, 4 quarters = 1 dollar, 5 nickels = 1 quarter, and so forth. Charts of these handy equivalences should be posted in classrooms and referred to daily.

7. Recent studies have shown that under the right conditions, young children develop largely self-taught informal approaches to calculate and solve problems. If children are given frequent opportunities to work with concrete material such as money, base 10 blocks, number charts, and measuring devices, they often will develop informal methods of calculation incidentally without explicit instruction. An example is: finding the difference between a length of 78 cm and 100 cm on a metre stick by reasoning "2 more centimetres equals 80 cm and another 20 centimetres makes 100 cm. So the difference equals 22 cm."

Games that involve calculations in the head should be used regularly to stimulate student thinking. In the early primary grades, dice games such as Snakes and Ladders require students to reason out sums and count on the game board. Games such as Monopoly involve working with place value relationships. Anyone who has played darts knows that figuring in the head is as important a skill to have as good hand-eye coordination. Teachers can construct ring toss games having the same rules to eliminate the need for darts, an obviously potentially dangerous classroom material.

8. Children should learn to equate a variety of expressions involving quantitative language. They need to realize that some terms are rarely seen in written form while

others such as symbols are exclusively written. One rarely sees the expressions "8 minus 4, 16 plus 8, or 4 times 12." Conversely, we cannot "say" the symbols $+$, $-$, $\times$ or $\div$. We have to interpret these using some form of oral expression.

We also should ensure that students are exposed regularly to the word forms of numbers: for example, "add fifty and one hundred twenty five." In the typical classroom, students rarely hear numbers because they work almost exclusively with written computation. As a consequence, children do not learn how to read and interpret numbers because these skills are not required in carrying out written calculations. Students are less likely to reach for a paper and pencil to work out 25×1000 if they say to themselves "twenty five times one thousand is the same as 25 thousands. So the answer is 25 000."

9. Children should understand that there is no best method of calculating mentally. Unlike written calculations that usually involve only a single approach, the approach one takes to calculate mentally depends entirely on the types of numbers as well as the situation. For example, one person might use front-end addition to calculate the sum of 75 and 75 by reasoning "70 and 70 equals 140; 5 and 5 equals 10; so the answer is 150," while another person might think "Well, I know that 75 cents and 75 cents equals \$1.50; so the answer must be 150."

10. A final suggestion: As several studies have suggested, the ability to do even straightforward mental calculations should not be taken for granted. It is surprising how many children believe that they need a paper and pencil or calculator to work out simple calculations such as $250 + 250$ or 22×1000 because the "numbers are too big." Teachers need to show the less able students that they have the ability to work out such calculations in the head if they use what they know about numbers.

A List of Mental Calculation Strategies with Accompanying Examples

In the list that follows, each strategy is briefly described and clarified through the use of a few examples. It is assumed that these strategies would be taught in a meaningful way using a variety of appropriate concrete materials and the instructional approach discussed earlier. The strategies can be adapted for various grade levels simply by changing the size and type of numbers. I have tried to identify the most commonly seen strategies. However, because there are many strategies that other writers might include, the list should not be considered complete.

1. Counting on and back

Adds and subtracts by counting on from one addend or back from the minuend in steps of 1 or greater.

$6 + 2 = ?$	Count 2 more from 6 : 7, 8. The answer is 8
$3 + 49 = ?$	Count 3 from the larger addend 49:50,51,52.
$147 + 3 = ?$	Count 3 from 147.
$8 - 2 = ?$	Count back 2 from 8.
$998 - 3 = ?$	Count back 3 from 998.
$26 + 8 = ?$	Count on from 26 by 4 and then 4 again.
$125 - 6 = ?$	Count back from 125 by 5, and then by 1 more.
\$1.55, change from \$10	5 makes 60, 40 makes \$2, \$8 more equals \$10. Change equals \$8.45

2. Bump-up and bump-down

Adds and subtracts by considering the endings of the numbers.

$26 + 8 = ?$	Since $6 + 8 = 14$, last digit is 4, the decade 2 is "bumped up" to 3. The answer is 34.
$166 + 5 = ?$	Since $6 + 5 = 11$, last digit is 1, and the decade 16 is increased by 1. The answer is 171.
$123 - 4 = ?$	Since $9 + 4 = 13$, the last digit must be 9. Decrease the decades by 1. The answer is 119.

3. Annexing 0s to add and subtract

Adds and subtracts with multiples of 10, 100, and 1000.

$10 + 10 = ?$	1 ten and another 10 equals 2 tens or 20.
$200 + 300 = ?$	2 hundred plus 3 hundred equals 5 hundred or 500.
$6000 - 2000 = ?$	Remove (annex) 0s. $6 - 2 = 4$. Replace 0s. The answer is 4000.
$45 + 30 = ?$	4 tens and 3 tens equals 7 tens or 70 and add back 5. The answer is 75.
$897 - 200 = ?$	8 hundred less 2 hundred equals 6 hundred and add back 97. The answer is 697.

4. Bridging

Adds and subtracts by alternating 10s and 1s.

$20 + 16 = ?$	$20 + 10 = 30$, 6 more equals 36
$43 + 39 = ?$	$43 + 30 = 73$, $73 + 9 = 82$.
$18 + 18 + 16 = ?$	$18 + 10 = 28$, $28 + 8 = 36$, $36 + 10 = 46$, $46 + 6 = 52$.
$4.3 + 2.8 = ?$	$4.3 + 2 = 6.3$, $6.3 + .8 = 7.1$
$200 - 36 = ?$	$200 - 30 = 170$, $170 - 6 = 164$

5. Adding and subtracting compatible numbers

Adds and subtracts by recognizing compatible numbers, numbers whose sum equals a multiple of 10.

$8 + 2 = ?$	I know that 8 and 2 equals 10.
$28 + 2 = ?$	Since 8 and 2 equals 10, the answer must be 30.
$75 + 25 = ?$	The sum is 100 because 3 quarters and another quarter equals 1 dollar or 100 cents.
$100 - 25 = ?$	Since 75 and 25 equal 100, the difference is 75.

6. Forming compatible numbers

Adds and subtracts by rearranging a sum or difference into recognized compatible numbers.

$8 + 5 = ?$	8 and 2 more equals 10; 10 and 3 equals 13.
$75 + 26 = ?$	$75 + 25 = 100$, $100 + 1 = 101$
$850 + 155 = ?$	$850 + 150 = 1000$, $1000 + 5 = 1005$
$5.5 + 2.6 = ?$	$5.5 + 2.5 = 8$, $8 + 0.1 = 8.1$

7. Adding and subtracting 8 & 9 endings

Adds and subtracts numbers ending in 8 or 9 by rearranging as a multiple of 10.

$9 + 7 = ?$	$10 + 7 = 18$, take away 1, the answer is 16.
$26 + 38 = ?$	$26 + 40 = 66$, take away 2, the answer is 64.
$53 + 99 = ?$	$53 + 100 = 153$, take away 1, the answer is 152.
$253 - 98 = ?$	$253 - 100 = 153$, add back 2, the answer is 155.
$\$1.98 + \$2.53 = ?$	$\$2.53$ plus 2 dollars equals $\$4.53$, less 2 cents equals $\$4.51$.

8. Front-end adding and subtracting

Adds and subtracts by working from the most to the least significant digits (left to right or front to back).

$$\begin{array}{r} 48 \\ + 56 \\ \hline \end{array}$$

$$40 + 50 = 90, 90 + 14 = 104$$

$$\begin{array}{r} 3.7 \\ + 2.5 \\ \hline \end{array}$$

$$3 + 2 = 5, 0.7 + 0.5 = 1.2, 5 + 1.2 = 6.2$$

$$\begin{array}{r} \$2.75 \\ + \$3.50 \\ \hline \end{array}$$

$$\begin{array}{l} \$2 + \$3 = \$5, 75 \text{ cents and } 50 \text{ cents equal } \$1.25. \\ \$5.00 + \$1.25 = \$6.25 \end{array}$$

$$\begin{array}{r} 100 \\ - 45 \\ \hline \end{array}$$

$$100 - 40 = 60, 5 \text{ from } 60 \text{ equals } 55.$$

9. Efficient skip counting

Adds and multiplies by skip counting in multiples of a factor or addend.

$$5 + 5 + 5 + 5 = ?$$

$$5, 10, 15, 20, \text{ and } 1 \text{ more equals } 21.$$

$$8 \times 5 = ?$$

$$\text{Count by } 5\text{s to } 40 \text{ or more efficiently by } 10\text{s.}$$

$$16 \times 50 = ?$$

$$\text{Count by } 50\text{s or better yet by } 100\text{s.}$$

$$0.6 + 0.6 + 0.7$$

$$\text{Count by } 0.6 \text{ to get } 1.8 \text{ and add } 0.1 \text{ more.}$$

10. Multiplying and dividing by annexing 0s

Multiplies and divides by annexing 0s to work with smaller numbers.

$$5 \times 30 = ?$$

$$5 \times 3 \text{ tens} = 15 \text{ tens. The answer is } 15 \text{ tens.}$$

$$7 \times 4000 = ?$$

$$\text{Annex } 0\text{s. } 7 \times 4 = 28. \text{ Replace } 0\text{s. The answer is } 28 \text{ } 000.$$

$$3500 \div 5 = ?$$

$$\text{Annex } 0\text{s. } 35 \div 5 = 7. \text{ Replace } 0\text{s. The answer is } 700.$$

11. Front-end multiplication

Multiplying by working from left to right.

$$\begin{array}{ll} 2 \times 25 = ? & 2 \times 2 \text{ tens} = 40, 2 \times 5 = 10, 40 + 10 = 50 \\ 4 \times 3 \frac{1}{2} = ? & 4 \times 3 = 12, \frac{1}{2} \text{ of } 4 = 2, 12 + 2 = 14 \end{array}$$

12. Multiplying 8 & 9 endings

$$\begin{array}{ll} 7 \times 99 = ? & 7 \times 100 = 700, 700 - 7 = 693 \\ 5 \times \$4.98 = ? & 5 \times \$5 = \$25, \text{ less 10 cents equals } \$24.90 \end{array}$$

13. Chaining

Completing a series of calculations non-sequentially by combining compatible numbers.

$$\begin{array}{ll} 12 + 17 + 18 = ? & 18 + 12 = 30, 30 + 17 = 47. \\ 99 + 56 + 1 = ? & 99 + 1 = 100, 100 + 56 = 156 \\ 4 \frac{1}{2} + 1 \frac{3}{4} + 1 \frac{1}{2} = ? & 4 \frac{1}{2} + 1 \frac{1}{2} = 6, 6 + 1 \frac{3}{4} = 7 \frac{3}{4} \end{array}$$

14. Halving and doubling

Multiplying by halving one factor while doubling the other until a stage is reached where another method can be employed. (A variation of factoring).

$$\begin{array}{ll} 12 \times 8 = ? & \text{half of } 8 = 4, \text{ double } 12 = 24, \\ & \text{half of } 4 = 2, \text{ double } 24 = 48, \\ & 48 + 48 = 80 + 16 = 96. \end{array}$$

$$\begin{array}{ll} 15 \times 12 = ? & \text{double } 15 = 30, \text{ half } 12 = 6, \\ & 6 \times 30 = 180. \end{array}$$

15. Factoring

Multiplying by rearranging the factors into more compatible ones.

$$\begin{array}{ll} 8 \times 8 = ? & 4 \times 2 = 8, 4 \times 8 = 32, \text{ double } 32 = 64 \\ 12 \times 15 = ? & 4 \times 15 = 60, 3 \times 60 = 180 \\ 16 \times 25 = ? & 4 \times 25 = 100, 4 \times 100 = 400 \end{array}$$

16. Aliquot parts

Multiplying by rearranging into compatible numbers and dividing to compensate.

$$12 \times 25 = ?$$

12 quarters. 4 quarters = \$1,

12 quarters = \$3, answer is 300.

$$\text{Or, } 12 \times 25 = 12 \times 100/4 = 1200/4 = 300$$

Footnotes

- ¹ The typical practice in the past is to teach all algorithms with fractions beginning around grade 6, repeat and attempt to remediate in grades 7 and 8, and finally ignore them in grades 9 to 12.
- ² When children in Desforges and Cockburn's study consistently made the same types of errors, the teachers repeatedly demonstrated the required routine rather than make an effort to diagnose the underlying problem. Reteaching the exact form of the algorithm is the typical corrective strategy in schools.
- ³ Many of these terms were conceived originally to describe actions on objects rather than on symbols. For example, the term "carry" referred to moving a counter from one column to another on an abacus. It seems inappropriate to use this term when describing symbolic actions. The word "calculation" is derived from the Latin word "calculus," the small stones or pebbles used as counters on an abacus.
- ⁴ There are a great number of games and activities that can be done on the calculator: for example, students can be given a calculator where the 9 key has been taped over. Their task is to figure out how to use the calculator with the "dead" 9 key to do calculations such as 5×989 , $909 + 589 + 199$, and so forth.
- ⁵ Although all human-produced calculations involve some mental processes, mental calculation typically refers to unconventional ways of computing in which writing aids are normally not used. In performing a mental calculation, perceived number relationships are often used to simplify a calculation.
- ⁶ Many students use this algorithm to calculate mentally. In conducting a study comparing skilled and unskilled mental calculators in the high school (Hope & Sherrill, 1987), it was found that these groups could be distinguished by their use of a mental analogue of the pencil-and-paper algorithm: To calculate mental products of two numbers, about 85% and 20% of the calculations attempted by the unskilled and skilled students respectively involved this mental analogue. One student in the study used it to calculate even the straightforward calculation 30×20 .

CHAPTER 10

ESTIMATION

The production of an "estimate" or "approximation" is an important mathematical and life skill that has been given scant attention in previous mathematics programs, perhaps because estimation seems out of place in a discipline well known for its exactness and precision. Considering the great numbers of articles written about estimation recently and the attention given by textbook publishers, estimation seems to be gradually assuming its rightful place in a revised mathematics program.

As with topics such as problem solving and mental calculation, estimation should not be taught separately from other topics in the mathematics curriculum. Rather it should form an integral part of most lessons dealing with problem solving, number and computation, and measurement.

Estimation is a complex process comprising several components as outlined in Table 1 below. The table is a modification of Sowder and Wheeler's (1989) analysis of the components involved in computational estimation. I have broadened their analysis to include the components of measurement estimation.

Conceptual Components

Role of Approximation

One of the major requirements in learning to estimate is to acquire an understanding of the terminology associated with the production of an estimate: e.g., terms such as "about," "estimate," "approximate," "close to," and "near." Students must be able to interpret the meaning of statements such as the following: "Nine boxes of candy contain 52 pieces of candy each. About how many pieces of candy are there in all?" Many young children do not understand the meaning of the term "about" and are very hesitant about using rounding to produce estimates, while older students don't feel comfortable in producing, what they see as, inexact answers. Sowder and Wheeler found, for example, that grade 7 and 9 students preferred the compute-then-round method (if this method can be called estimation) to the round-then-compute method. The researchers found that the students were often obsessed with accuracy and wanted to view both the exact and the estimated result so they could see how "far off" the estimate was. They concluded that even as late as the ninth grade, "young adolescents do not recognize that an essential component of computational estimation is computing with approximate numbers" (p. 139).

Table 1. An Analysis of the Components involved in Computational and Measurement Estimation. Based on the work of Sowder and Wheeler.

I. Conceptual Components

1. Role of Approximate Numbers

- 1.1 Recognition that approximate numbers are used to compute and that all measurement is approximate
- 1.2 Recognition that an estimate is an approximation

2. Multiple Processes/Multiple Outcomes

- 2.1 Acceptance of more than one process for obtaining an estimate
- 2.2 Acceptance of more than one value as an estimate

3. Appropriateness

- 3.1 Recognition that appropriateness of process depends on context
- 3.2 Recognition that appropriateness of estimate depends on desired accuracy

II. Skill Components

- 1.1 Reformulation: Changing the numbers used to compute
 - 1.1a Rounding
 - 1.1b Truncation
 - 1.1c Averaging
 - 1.1d Changing the form of a number
- 1.2 Compensation: Making adjustments during or after computing
- 1.3 Translation: Changing the structure of the problem

III. Related Concepts and Skills

Computational Estimation

- 1. Powers of ten
- 2. Place value
- 3. Number comparison
- 4. Mental computation
- 5. Basic number facts
- 6. Number properties and their use
- 7. Influence of number modification

Measurement

- 1. Unit comparison
- 2. Basic numeric equivalents
- 3. Benchmark equivalences

IV. Affective Components

- 1. Confidence in ability to do mathematics
 - 2. Confidence in ability to estimate
 - 3. Tolerance for error
 - 4. Recognition of estimation as useful
-

Students need to recognize that all measurements are approximations regardless of the precision of the measuring instruments. Analyses of current teaching practices and textbooks have identified many unrealistic and misleading ideas about estimation and precision. For example, in some textbooks, lengths of objects are pictured as having a whole number of units even though they are accompanied by the question, "About how long is each object?" In the real world of measurement, exact whole-number results are highly unlikely. One textbook lesson identified by Kaster (1989) includes the following presentation:

Birds fly south for the winter each year. The bobolink travels about 6500 kilometres. The blue goose travels about 2481 kilometres. About how much farther does the bobolink travel?

Positioned below this statement was a step-by-step illustration of the written procedure for finding the difference between 6500 and 2481. The text page concluded with the answer, "The bobolink travels about 4019 kilometres farther than the blue goose." At least two problems exist with this presentation. First, the measurement of 6500 km suggests that it was made to the nearest 100 kilometres (i.e., a distance between 6450 km and 6550 km) or it was simply an estimate, while the measurement of 2481 km implies a precision of 1 kilometre (i.e., a distance between 2480.5 km and 2481.5 km).¹ It is clearly nonsensical to subtract measures of different precision. Second, students are being taught implicitly to consider calculation with untransformed data as a legitimate form of estimation and, once again, being shown that measurement performed in school bears little relation to the real world.

In the elementary program, the terms associated with estimation should be introduced as early as year 1. If students are to appreciate estimation, they should be encouraged to report intervals rather than unique answers: for example, "My desk measures between 60 and 61 cm. It looks closer to 60 cm though." To encourage children to consider ranges of measurement data, comparisons such as the following should be regularly presented: for example, knowing that a sum \$1.79 and \$2.87 must be more than \$3 but less than \$5. These types of activities may help students understand that computation with approximate numbers yields approximate answers, a legitimate mathematical activity.

In the middle grades, students should be provided with the following types of estimation activities to help them acquire a "feel" for the relative size or position of fractions and decimals.:

- On a number line containing the labels 0, $\frac{1}{2}$ and 1, show the approximate position of the following fractions: $\frac{1}{4}$, $\frac{2}{3}$, $\frac{7}{8}$, and $\frac{15}{16}$.

- If an eyedropper full represents one millionth, estimate the size of the whole unit.

Students of all ages should be challenged to provide estimates of quantities not readily determined: the population of the school, the number of hamburgers spiced by a 1 litre jar of ketchup, the population of the province, the number of 12-year-olds presently alive, and so forth. Students can be encouraged to explore the various ways that such information can be obtained.

Students should also learn how to distinguish exact from approximate numbers when placed in a context. For example, which of the following statements likely contain exact or approximate values:

- The population of Saskatchewan dipped under 1 million.
- A lottery winner earns over \$25,000.00 monthly.
- The TSE index dropped 153 points this week.
- The GST will be assessed at a 7% rate.
- Michael Jordan scored 55 points against the Celtics.

Multiple Processes and Multiple Outcomes

Children also need to learn that different estimation procedures and different approximate values can be used to produce legitimate estimates as illustrated below.

<u>Problem</u>	<u>Estimate 1</u>	<u>Estimate 2</u>
Estimate the sum of 27 and 49	$30 + 50 = 80$	$50 + 25 = 75$

In Sowder and Wheeler's study of estimation performance, third graders were found to be hesitant to accept two different ways of rounding the same number, primarily because of their lack of experience with rounding. Although older students are more accepting of multiple processes, they are less accepting of multiple answers, probably because mathematics problems in school typically have a single, correct answer.

Teachers will have to encourage children to find several ways to produce estimates and to resist the urge to have children find the "best" estimate because calculations on the actual numbers produce the "best estimates." Practical experience indicates that children tend to calculate with the actual numbers (either mentally or with paper and pencil) and round to produce, what they believe is, an estimate.

Appropriateness of an Estimate

The desired accuracy of an estimate depends entirely on its use. Whether to produce a rough estimate, fine estimate, or an exact answer depends on the ultimate use of the estimate: clearly Revenue Canada would not be satisfied with an estimate of our net incomes, for example. Because no clear-cut rules can be invoked, children will have to be provided with considerable help and practice in determining the desired accuracy of an estimate.

We should try to be realistic in discussing estimates. For example, most adults would be content with an estimate of \$100.00 ($5 \times \$20$) for determining the cost of 5 shirts priced at \$18.98 each. Few would prefer or require the closer estimate of \$95 ($5 \times \$19$); fewer still would desire the exact answer \$94.90. If we want to know whether \$10 is enough money to purchase three items priced at \$1.59, \$3.86, and \$2.55, then rounding to \$2, \$4, and \$3 respectively provides us with the information to answer our question. However, if the items are priced at \$3.86, \$4.35, and \$1.86, then we have to obtain a more precise estimate to answer our question, especially if we consider sales tax. (Incidentally, \$10 is not enough.)

In textbooks, the importance of context in determining estimates is largely ignored. Most calculations are done in the absence of a context: e.g., multiply 12.5 and 4.35; estimate the product of 4.38 and 5.67. When a context is provided it usually contains unrealistic numbers, portrays an unrealistic situation, or is used to produce unrealistic answers. One textbook page pictured two trees accompanied by the following statement:

The Grizzly Giant Tree is about 64.3 metres tall. The General Sherman tree is about 1.3 times the height of the Grizzly Giant Tree. About how tall is the General Sherman Tree? (Kastner, 1989, p. 44)

The text illustrated a calculation of 64.3 by 1.3 and concluded with the statement, "The General Sherman Tree is about 83.59 metres tall." Unfortunately, textbooks frequently use contexts such as these to demonstrate mathematical procedures rather than to illustrate interpretative thinking.

Because context is generally ignored in school or is used to promote algorithmic responses, children do not analyze their answers carefully. For example, when 13-year-olds in an American survey were given the problem below, about one third expressed their answers using decimal or fractional remainders (31.33 or $31 \frac{1}{3}$):

An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?

Nearly a quarter of the pupils ignored the remainder altogether and presumably were content to let 12 soldiers walk or commandeer their own transport.² Until textbook presentations are more realistic and designed to encourage interpretive thinking, teachers should use a meaningful context to teach estimation, typically involving some measurement situation in which estimates would be preferred to exact answers. It is also recommended that students be encouraged to uncover the "old-fashioned," unrealistic material in textbooks and change such examples or problems to bring them into the modern world (Kastner, p. 46).

Skill Components

Estimation also involves a number of process and procedural skills that will have to be explicitly taught to students, although most able students seem to acquire them naturally without explicit instruction.

Researchers such as Sowder and Wheeler have identified three computational estimation processes; each process has been modified slightly to include measurement estimates: (1) reformulating the estimation task by changing the numbers/measures and operating on these reformulations; (2) compensating by making adjustments during or after computing; (3) translating by changing the structure of the estimation task.

Reformulation

Simply put, reformulating a problem or task calls for replacing the original numbers or measures before computing or reasoning. **Rounding** numbers is perhaps the most popular form of reformation, especially since it is likely the only one taught in school. Unfortunately, when rounding is taught in school, it is rarely done for a purpose. Young children find rounding difficult because (1) they must decide which value to round to (should 1234 be rounded to the nearest 1000, 100 or 10?), and (2) they must identify the closer multiple of a power of ten (is 1000 or 2000 closer to 1489?).

Truncating numbers by eliminating ("chopping off") the right-most digits is conceptually more straightforward but, in most instances, less accurate reformation process than rounding because it provides an underestimate:

123	Examine left-most digits and add:
469	$1 + 4 + 2 + 4 + 3 = 14$.
265	Estimate is 1400.
425	
+ 388	

Children can learn to improve the estimate by combining the numbers represented by the remaining right-most digits. For example, in the preceding computation, we

might estimate that 88 and 23 equal 100, 65 and 25 equal 100 and add 200 - or say 250 to add a little bit extra - to produce a good estimate of about 1650.

Researchers have found that students' ability to round or truncate varies with age. Few grade 3 students can use rounding effectively to estimate computations such as the following: $290 + 170 + 110 + 220$. Because truncating alone does not usually produce an estimate as accurate as rounding, older students consistently prefer rounding. Rounding should not be presented until at least grade four or five and, when it is presented, it should be taught as a method of estimation.

Averaging is an estimation procedure that is best applied when the numbers cluster around a constant: e.g., estimating the sum of 39, 36, 44, 38, and 41 by reasoning that all addends are close to 40 and, consequently, $5 \times 40 = 200$. Studies have shown that, although students accept averaging as an "okay" method, very few claim they would use the method to produce an estimate.

Changing the form of a number involves the substitution of an equivalence to simplify a calculation: e.g., $79 \times \$0.51$ is calculated as 80 half dollars or \$40.00. Elementary and middle years students find this method to be very difficult. However, they might have less difficulty if we presented simple examples right from the beginning: for example, 8×6 cents is more than 40 cents because 8 nickels equals 40 cents; $6 \times 3/4$ cup must be more than 3 cups because 6 half cups equals 3 cups.

Compensation

Compensating as shown earlier and illustrated in the following example is often done to increase the accuracy of an estimating procedure such as truncation:

Problem

Estimate the sum of
\$3.20, \$2.70 and
\$1.30.

Reasoning

$\$3 + \$2 + \$1 = \$6$
Plus one more dollar
equals \$7.

Compensating can be difficult for some children because its use depends on the context of the estimation problem and requires children to find "compatible numbers."

Translation

Translating brings about a change in the structure of a problem: e.g., changing division to multiplication. In Sowder and Wheeler's study (1989), grade 7 and 9 students were presented with the computation 564×78 and shown the following approach to used to develop an estimate: " $1000 \times 40 = 40\ 000$, + 2000 more, so 42 000." The strategy

involves the "halving and doubling" mental calculation strategy discussed in Chapter 9 plus some additional compensation. The students claimed the procedure was too complicated or believed it involved guessing.

Related Concepts and Skills

A number of concepts and skills underlie computational estimation. Although many might be taught and learned separately, some are best learned while learning to estimate.

Powers of 10

Obviously, students need to be able to work with powers of ten (adding, subtracting, multiplying and dividing) and have a good understanding of place value (reading, writing, rounding, truncating). Few textbooks include instruction on truncation and, although most students learn to round numbers, they refuse to consider alternatives to the school-learned procedures. For example, Sowder and Wheeler found that many grade 5 and 7 students when presented with the estimation problem $267 + 519$ objected to the very good estimate $250 + 500 = 750$. They argued that the number 300 must be used instead of 250 because "You're always taught to go up if it's past five," or "Seven is above five, so you have to go up not down."

Comparison

Students need to be able to compare numbers and determine the closest number to a given number. Elementary children need regular practice with activities such as the following:

1. Which number is closest to 100? 85, 90, 130
2. Which amount is closer to \$5.00? \$4.50, \$5.25
3. Which height is closest to 10 m? 850 cm, 1050 cm

Mental Computation and Recall of Basic Facts

Many studies have demonstrated that good estimators often make use of mental computation including recall of the basic number combinations to increase the accuracy of their estimates. Reys, Rybolt, Bestgen, and Wyatt (1982) found that proficient computational estimators could be characterized by:

... the quick and efficient use of mental computation to produce accurate numerical information with which to formulate estimates. All estimators exhibited well-developed skill with multiples of 10 or a limited number of digits, and many others were fluent in mentally computing with larger numbers, more digits, and even different types

of numbers (e.g., fractions). On some problems, subjects resorted to mental computation rather than using an estimation technique. For these problems and these students, it was more efficient for the person to compute mentally rather than estimate. (p. 197)

Number Principles

In order to become proficient computational estimators, students need to understand and apply many different number properties such as those listed below (for brevity, the principles are described using symbols rather than words:

<u>Principle</u>	<u>Application</u>
a b, c d then $a + c$ b + d	$89 + 49$ can be estimated as $100 + 50$. 150 must be an over-estimate.
a b, c d then $a + c$ b + d	$\$1.25 + \$2.03$ can be estimated as $\$1 + \$2$. $\$3$ is an under-estimate.
a b, c d then $a \times c$ b $\times$ d	69×79 can be estimated as 70×80 . 5600 must be an over-estimate.
a b, c d then $a \times c$ b $\times$ d	12 @ $\$1.25$ can be estimated as $10 \times \$1$. $\$10$ is an under-estimate.

Each principle can be taught using concrete aids, particular measuring devices, or by referring to calculations using smaller numbers.

Number Modification

A strength of good estimators is their ability to determine the effects on numbers when they're modified in some manner. For example, they know that a fraction decreases in size when the denominator is increased. Students in all grades should be provided with activities that require them to think about and explain the effect on the relative size of numbers when they are transformed in some manner. This form of number sense can be developed even in grade 1 by ensuring that children are given appropriate activities such as the following: identify the larger of $12 \div 6$ and $12 \div 7$ without calculating.

Unit Comparison

To help children improve their understanding of measurement notions, we should provide ample experience in unit comparison. In the primary grades, children should be able to estimate lengths that are shorter, longer, and close to a given length such as metre stick. In the upper elementary and in the middle grades, students should estimate which containers have about the same capacity as a unit container such as a litre container. Our everyday experience involves a great deal of such measurement

comparisons: "It's hotter and windier today than yesterday," "I ran about 4 kilometres today," and "He seems older than his wife."

Basic Number Equivalents

An important but often ignored aspect of estimation is being able to recall some basic numeric equivalents: for example, knowing the number of minutes in an hour, the hours in a day, the number in a dozen, the number of days in a week, the number of centimetres in a metre, and so forth. Children who do not know these equivalents will be limited in their ability to produce a reasonable estimate. For example, we cannot estimate the number of months in 100 days unless we know that approximately 30 days equals 1 month. There is ample evidence to suggest that students and many adults have not committed this useful information to memory. I am not suggesting that students be "drilled" in these equivalents. Instead, the students should be given regular opportunities to use these equivalents to determine estimates and to solve problems.

Benchmark Equivalents

Children also need to acquire a "feel" for measurement and learn many benchmark equivalents. We cannot estimate the length of a classroom unless we have a "feeling" for distances of 1 metre, 10 metres, and so forth. Special attention should be given to having children learn some commonly seen benchmarks. For example, they should know the approximate height of a doorway, the distance from home to school, and many of their own personal statistics such as their height and mass. Once acquired, they should use these benchmarks to help produce estimates: for example, "I know the dog's height cannot be 3 metres because the height of our classroom door is only 2 metres."

In summary, estimation is a considerably more complex process than most teachers and mathematics educators realize. Teachers will have to be given guidance in developing suitable activities to help children learn to estimate both inside and outside of school. Although formal instruction on the techniques of estimation should not begin until the later elementary grades, teachers in the primary grades should be laying the groundwork by ensuring that children acquire a good understanding of number and quantity. Sowder and Wheeler make the following observation:

This is not to say that nothing should be attempted in earlier grades - rather, just the opposite is true. Finding numbers close to a given value, recognizing which of two numbers is closer to a third, computing mentally, recognizing when a problem does not demand an exact answer, for example, are all topics worthy of more instructional time than is now accorded them. Without the careful development of the foundational concepts of computational estimation, instruction on computational estimation may result in the learning of a set of discrete, rote

algorithmic processes, distinct from a conceptual knowledge base. (p. 145)

Footnotes

- ¹ It's also unlikely that ornithologists have measured the blue goose's flight distance to this degree of precision. The issue of realism in textbook problems was discussed earlier in this report.
- ² Incidentally students who used calculators did worse on this item than those who used pencil-and-paper methods, indicating a need for teachers to help students interpret their calculations and check the reasonableness of their answers.

CHAPTER 11

MEASUREMENT

Studies of the everyday activities of the consumer and worker demonstrate clearly that a functional knowledge of measurement is extremely important, regardless of the advances in technology. Although few people actually carry out measurements on a regular basis, almost all are regular consumers of measurement information.

Measurement is strongly connected to other mathematical topics. Calculation, for example, usually makes sense only when the numbers being operated on represent some measurement units. In the real world, computations such as adding $1/2$ and $1/4$ are never done without considering units of measurement. Measurement is commonly used to provide a real world context for problems seen in school. Geometry and measurement are obviously interconnected and support each other in many ways. Most number and numeration notions (counting, fractions, decimals, whole numbers, place value) can be represented by reference to measurement and, consequently, a study of measurement can reinforce students' understanding of these quantitative notions and contribute to the learning of mathematics in other areas.

Success in many quantitative disciplines such as the physical sciences depends on a good understanding of the measurement process and a knowledge of specific measures. Students will not be able to participate actively in our measurement-oriented society unless they know something about the nature of measurement and its role in advancing knowledge in science and technology.

Many measurement ideas are learned incidentally as we carry out our everyday tasks. Carpenters, for example, likely acquire more knowledge about linear measurement while on the job than in school. But the school has a responsibility to ensure that children eventually develop an understanding of the nature of measurement as a system as well as acquire specific knowledge concerning the common measurements of length, area, volume, capacity, temperature, time, mass (weight), and angles. Thus, measurement instruction should begin in grade 1 and continue throughout every grade.

The Measurement Process

How do children learn to measure and what should they learn about measurement? In a comprehensive program, children need to learn to acquire the following understandings about the measurement process: (1) measurement involves the perception

of an attribute; (2) measurement involves comparisons; (3) measurement involves quantification, the application of standardized procedures using appropriate measurement tools. These understandings can only develop through a mathematics program that involves children in meaningful measurement activities (see Inskeep (1986) for a discussion of other measurement goals).

Measurement as Perception

At its most basic level, measurement involves the quantification of a perceived attribute (Inskeep, 1986). In order to measure the area of a table, for example, students need to be able to perceive area before this property can be measured.

Moreover, students need to learn how to distinguish between different types of measures: the difference between area and volume, for example. Consequently, teachers need to involve children in activities designed to develop their perception of measurable attributes, especially in the beginning stages of learning to measure.

Measurement as Comparison

All measurements involve a comparison of attributes. Objects can be compared on any number of dimensions: their mass, weight, area, volume, temperature, monetary worth, sharpness, softness, beauty and so forth. Even nebulous but commonplace expressions such as "He's a nice person" presupposes that some comparison can be made on the "niceness" scale and that people can be compared on the degree to which they are "nice." The social sciences have created a major industry out of quantifying intangible human qualities such as intelligence, leadership, personality and so forth.

Much of our everyday language involves comparisons: "the taller boy," "the heavier man," "the faster runner," and so on. Children need to be regularly exposed to the language of measurement comparisons including the many metaphors involving measurement ideas: "slow as molasses," "a stitch in time saves nine," and so forth.

Even when a measurement is expressed as a quantity, an implicit comparison is made. For example, a measured length of 30 cm implies a comparison between the length being measured and a standard length of 30 one-centimetre units. Although measurement comparisons typically involve numbers, comparisons are frequently made in the absence of any quantification. For example, the dimensions of the Queen Mary (the ship, not her royal majesty) do not need to be known to conclude that it is larger than a dinghy. Much of the introductory work with measurement should involve comparisons before students learn to work with conventional measurement tools and standard units.

Measurement as quantification

Eventually, children must be able to perform standardized procedures with commonly seen measurement tools. Many educators (see Inskeep, 1976, for example) recom-

mend that children should be introduced to quantification using non-standard rather than standard units: e.g., measuring the length of a classroom by pacing rather than by using a tape measure. In this way, children are able to attend to the process of measurement without being distracted by the unfamiliarity of a standard unit. As Inskeep suggests, "Beginning with metres, litres, and grams may sound good to the adult but may not be significant to the child" (p. 78). When this preparatory work is complete, children should be exposed to standard units, their symbols, and the associated numerical relationships (e.g., 1000 m = 1 km).

Judging from the results of the 1985 British Columbia Mathematics Assessment (Robitaille & O'Shea), many children are still more comfortable working with Imperial units than with SI units, despite official policy since 1976 that all teaching materials developed in Canada use metric units. Table 1 contains some sample items and results of the study common to grade 4 and grade 7 students (the numbers in the results column represent percentages, and the bold-emphasised response represents the correct answer):

Table 1. Sample Items and Results from the 1985 B.C. Mathematics Assessment

Item	Choices	Results	
		Grade 4	Grade 7
A ten-year-old boy is likely to weigh	A. 35 grams	4	4
	B. 75 grams	14	8
	C. 35 kilograms	23	51
	D. 75 kilograms	39	27
	E. Don't know	19	10
The temperature on a sunny summer day would most likely be	A. 5 Celsius	3	3
	B. 25 Celsius	41	50
	C. 55 Celsius	24	20
	D. 85 Celsius	21	21
	E. Don't know	10	5

Other items and results are listed below:

1. 67% of grade 4s selected metres as the unit used to measure a house given the choices mm, cm, km, and I don't know.
2. 61% of grade 4s and 76% of grade 7s correctly completed the statement "5 metres is the same length as" given the choices 50 centimetres, 500 centimetres, 50 millimetres, and 500 millimetres.
3. 38% of grade 7s correctly answered the question "250 g is how many kilograms?". The choices were 25, 250, 0.25, 2.5, and I don't know.
4. Only 44% of the grade 7 students knew that the prefix "centi" meant 1/100 of the unit of measure.

The students also completed a questionnaire that contained some questions that indirectly assessed which measurement system they preferred to use outside of school. The results indicate that many students still prefer to use Imperial units of measurement. Grade 7 students had a stronger preference than grade 4 students, despite three more years of supposedly metric instruction in school. Table 2 contains the items and results. The instructions were:

Both answers for each question are correct. If you were asked each question, which one of the two answers comes to your mind first?

Table 2. Items and Results Related to Student Measurement Preference of Grade 4 and Grade 7 Students in the 1985 B.C. Mathematics Assessment

Question	Choices	<u>Percent choosing</u>	
		Grade 4	Grade 7
1. How much does a bicycle weigh?	A. About 15 kilograms	41	34
	B. About 35 pounds	47	56
2. What is the temperature in this room?	A. About 70 degrees	21	35
	B. About 20 degrees	66	54
3. How far is it from Prince George to Prince Rupert?	A. About 700 kilometres	37	36
	B. About 450 miles	50	53
4. How much gasoline can the gas tank in a large car hold?	A. About 20 gallons	42	46
	B. About 90 litres	44	43

Note: The report did not provide an explanation for the totals not being close to 100%.

Assuming that students in Saskatchewan are similar to students in British Columbia, the results are disappointing. Do the results indicate some fault with our teaching approaches or simply reflect the current reality of living in a society where both Imperial and SI units continue to be used? A study should be done to determine the degree to which Imperial units continue to be used in school and to identify commonly used teaching practices.

In a revised elementary and middle years mathematics program, children need to learn the basic ideas underlying length, area, volume and capacity, time, mass, temperature, and angles. As in the other areas of the program, an active approach should be emphasized and children should be involved as much as possible in meaningful activities that emphasise the carrying out of actual measurements and the analysis of this gathered quantitative information.

The Understanding of Length

Before students are introduced to linear measurement devices such as rulers, tape measures, and so forth, teachers need to ensure that the attribute of length is clearly

understood by engaging students in introductory activities such as tracing lengths of lines drawn on the board or pacing off the length of the classroom. At this stage, the teacher should ensure that children perceive the attribute of length.

During this initial stage, both informal and formal terminology should be introduced and the children should be encouraged to demonstrate their understanding of these terms by applying them appropriately. Terms such as the following should be used: distance, height, width, length, depth. Gradually students need to be introduced to more specialized terminology such as perimeter, circumference, radius, diameter, diagonal, and so forth.

Analyses of the 1986 National Assessment of Educational Performance (see Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988, for example) indicate that students do not adequately understand this terminology. For example, one NAEP item illustrated a figure of a rectangle and the indicated length and width of 7 and 4, respectively. Students were asked the question, "What is the perimeter of this rectangle?" Only 17% of grade 3 students and 46% of grade 7 students answered correctly. Surprisingly, performance **decreased** on a parallel item in which the figure was absent but the term "perimeter" eliminated. Students were asked the question, "What is the distance around a 4 by 7 rectangle?" Only 15% of grade 3 students and 37% of grade 7 students answered correctly. Approximately 74% of grade 7 students could identify the diameter of a circle, yet only 44% could identify the radius. Such findings suggest that teachers need to spend more time on developing the understanding of specialized terms including their underlying meanings and to use them frequently in other mathematics lessons and in other subject areas. Teachers should be provided with a comprehensive list of comparison and measurement terminology and given some guidance as to which terminology should be emphasised at a particular grade level.

Linear comparison can be taught relatively easily to most children by having children directly compare lengths and distances: For example, comparing the different heights of children in the class and comparing these heights to other objects in the surrounding environment such as the height of the door. The following types of terminology should be introduced in context: taller than, shorter than, longer than, shorter than, wider than, and so forth.

As children become confident, they should be challenged to solve simple comparison problems involving imagined rather than seen objects: for example, "Who is taller, a giraffe or a horse." These activities lay the foundation for estimating length. Estimation made with a standard unit present can help children establish a mental image of that size prior to its actual use in measurement: for example, asking children to identify common objects in the classroom shorter than, longer than, and about as long as a displayed metre stick.

Teachers need to be aware that comparisons often involve transitive reasoning beyond the capabilities of some young children.¹ Steffe and Hirstein (1976), in analysing the work of Piaget, report that such reasoning develops gradually. Preschool children make comparisons visually without necessarily using an direct referent. Children in the age range of five to seven years use manual transfer; they physically move objects close together before making a visual comparison. Steffe and Hirstein note that children in this age range begin spontaneously to use their bodies to make comparisons (p. 42). For example, they will measure a length with their outstretched hands and then compare this length with a third length. Children in this stage are beginning to reason transitively - comparing lengths to a referent length - but such reasoning is intuitive and not well formed. Children aged eight years and over typically compare lengths to a referent length. That is, they can compare the heights of two towers by using a middle or third rod.

Textbooks typically promote only visual comparison. A prototype lesson illustrates two objects aligned perfectly: i.e., two crayons are pictured with each left end vertically aligned. Textbook lessons with their stereotypic illustrations generally contribute little to a children's understanding of measurement and can often inhibit their developmental growth.

Eventually children need to quantify lengths by measuring them in a variety of ways. Initially, measurement should involve non-standard measures such as the following: body units (foot, hand, cubit, paces, etc.), common classroom objects (pencils, paper clips, strings of given length, straws, etc.). Later instruction should involve conventional tools and standard units. Schools should be well equipped with a variety of linear measurement tools: metre sticks, cloth and metal tape measures, calipers, trundle wheels, feeler gauges and so forth. A micrometer should be used in the middle years to provide students with experience in measuring fine distances, the thickness of a human hair, for example. It also provides practice in reading scales and using decimal notation. In a revised program, students should be regularly measuring "something" with "something." Teachers will have to be provided with good measurement materials and some guidance on how to use them.

Teachers should emphasize that the choice of a measurement device depends entirely on the measurement problem. For example, a 30-cm ruler is well suited for measuring small straightline distances, but it is highly unsuited for measuring either long distances or fine lengths and the lengths of curved objects. I believe it is important for children to be explicitly taught some of the "folk wisdom" that adults use to measure lengths: measuring a length with a ruler but not necessarily from an endpoint; measuring the circumference of a sphere by wrapping string around the sphere and then measuring the string against a ruler; and so forth.

The Understanding of Area

The term area may be defined as the amount of surface contained within a planar region. If children are to understand that area involves the covering of a plane surface, activities centering on the covering or tiling of surfaces such as the following should be emphasised: (1) forming mosaics or tessellations using a variety of different figures; (2) covering solid figures with paper to determine the surface area; (3) covering plane shapes with common body objects such as hand or thumb prints; (4) tracing and cutting out imprints of plane objects and the surfaces of solid figures.

Many of these activities can be used to help children distinguish between the perimeter of figures and their corresponding areas. Kouba et al. report that about one third of the NAEP grade 7 sample confused area with perimeter. Students also had problems calculating the area when presented with a picture of a rectangle labelled with the dimensions of 5 and 6. The question was answered correctly by 20% of grade 3 students and 56% of grade 7 students. NAEP obtained a seemingly odd result on a question requiring the students to find the area of a rectangle drawn with 5 rows of 6 squares. The question was answered by only 5% of the grade 3 students and only 46% of the grade 7 students. These results suggest that students simply look for numbers to use in a memorized formula and do not possess a sound conceptual understanding of measurement. Similar results were obtained in the 1985 B.C. Mathematics Assessment. Only 31% of grade 7 students correctly calculated the area of 42 when shown a picture of a right triangle with the base and height labelled as 14 and 6, respectively. About 39% chose the response 84, the product of 14 and 6.

Students often are presented with stereotypic area "problems" where only the required dimensions are given. Consequently, they experience considerable difficulty with non-standard situations. For example, grade 7 students participating in the B.C. Assessment were asked to calculate the area of a figure bounded by two rectangles - the area between two superimposed rectangles (imagine a rectangle with a rectangular "hole.") All necessary dimensions were given and, to solve the problem, the student merely had to calculate the difference in area of the inner and outer rectangles. Only 30% of grade 7 students could solve this problem. The most popular choice (27%) was the area of the inner rectangle.

Children in the initial stages of learning frequently confuse area with other measures such as length: for example maintaining that the surface with a longer boundary is greater in area than a larger surface with a shorter boundary. Piagetian researchers have demonstrated that most younger elementary school children lack the ability to think about more than one relationship simultaneously, making it difficult for them to coordinate measurements such as area having two dimensions each of which affect the amount of area.

Children who cannot conserve area do not understand that the area of a surface is unrelated to its shape and that the area remains invariant with respect to a rearrangement of a surface. For example, younger students will erroneously conclude that the area of a square cut on its diagonal and rearranged to form a triangle is less than the area of the original square.

Children need to learn to compare the areas of surfaces in a variety of ways. They should first learn to superimpose surfaces that can be imbedded without overlapping any boundary to conclude that the superimposed surface has less area than the other surface. They should next learn to cut and rearrange superimposed surfaces that do overlap on some boundary. Once children are comfortable with the notion of comparing by superimposing, they should be taught the measurement of area by unit iteration: for example, counting the number of unit squares needed to cover a surface. It should be emphasised that children should experiment with a number of unit shapes before they are introduced to such standard measures as square centimetres, square metres, and so forth.

In the elementary and middle grades, children should not be forced to learn and memorize formulas they typically do not understand. In the current elementary program, students are expected to learn the following formulas by the end of year 5, the last year of the elementary program: (1) calculate the area of rectangular polygons by use of the formula $A = l \times w$; (2) using the concept that the area of a right angled triangle is one half that of a rectangular polygon to calculate the area of a right triangle. Unfortunately, children who memorize formulas without understanding typically apply them incorrectly. As the results of the NAEP and B.C. Assessment studies indicate children who memorize area formulas usually equate area with multiplication and blindly apply the rule to any surface where measurements are provided.²

Instead of presenting formulas to children, we should encourage them to work out various strategies for calculating the areas of regularly shaped objects: for example, forming rows of unit squares to calculate the area of a rectangle or cutting and rearranging a parallelogram to form a rectangle whose area can be easily determined. Such activities can serve as a good basis for the development of formulas in the later middle years.

The Understanding of Volume

Children need to perceive volume and its counterpart, capacity, as a measure of space.³ Children in the elementary grades find the concept of volume to be very abstract and difficult to comprehend. In fact, Piagetian researchers claim that before the adolescent years, most children are capable of developing only a limited understanding of volume. Naturally, children are confused by the three-dimensional

aspects of volume (Steffe & Hirstein, 1986, p. 47). For example, few elementary school children understand that various constructions differing in shape can be equivalent in volume. For example, when children are shown or build various rectangular prisms constructed from 36 unit cubes ($4 \times 3 \times 3$, $12 \times 3 \times 1$, $2 \times 3 \times 6$, and so on), they usually conclude that their volumes differ, **even though they know each contains the same number of unit cubes**. They experience even more difficulty when they are required to construct forms having equivalent volumes: for example, building a "house" of unit cubes on a 2-unit by 2-unit base having the same number of "rooms" as another form built on a different sized base. Indirect methods of measuring volume such as immersing objects in water and measuring the displaced volume are clearly beyond the intellectual grasp of elementary school children and should be presented in the later middle years.⁴

Children in the elementary grades should be involved primarily in activities that encourage the filling of objects with fluid-like materials or the packing of containers with solid objects instead of in the more formal pencil-and-paper work typically done in the current program:

- filling cans, bottles and various containers with cupfuls, handfuls, boxfuls, spoonfuls of sand, vermiculite, grain, water, and so forth.
- packing boxes with wooden cubes or other smaller boxes or filling cans with marbles, balls, and other non-tessellating materials.
- building and dismantling constructions made from wooden cubes or multi-link cubes.

With sufficient experience with these intuitive activities, children can be gradually led to the use of standard units such as cubic centimetres for volume and litres for capacity. As in the present program, most work should focus on units of capacity rather than volume. Again, as with area, formulas should not be introduced at the elementary level. Formulas for volume should be developed concretely through the building and analysing of constructions. As with their understanding of area, students do not acquire a complete understanding of volume and tend to think that it means multiplying any three dimensions of a solid. They also confuse surface area and volume.

It must be emphasised that volume and capacity concepts are best learned by doing rather than reading some textbook treatment. As will be discussed in the chapter on geometry, children have difficulty in relating the textbook diagrams of such figures as rectangular prisms to actual three-dimensional objects. For example, in a recent textbook written for grade 5, volume is introduced through pictures of rectangular prisms whose volumes are calculated by constructing layers of unit centimetre cubes,

leading to the formula length $\times$ width $\times$ height. Then students are presented exercises on calculating volumes of aquariums by multiplying the three specified dimensions. The text devotes only 5 pages to the treatment of volume.

The Understanding of Mass

The mass of an object can be sensed directly only through the lifting and holding of objects.⁵ Before children enter school, they will have heard the terms "lift," "push," "pull," and "heavy." Most children learn these terms in the context of some physical task involving muscular activity. Yet most teachers do not appreciate the degree to which the concepts of mass and weight are derived through, what Lovell (1971) terms, muscle sense. Instead, they rely almost entirely on the indirect presentations of workbooks and textbooks rather than "hands-on" activities. In the teaching of mass, the best way to use a textbook is to let the children weigh it.

Teachers must recognize that perception frequently interferes with the understanding of mass. Children up to about the age of 9 typically believe that spatial changes influence the mass of objects. For example, in a classical Piagetian experiment, children were asked to form two balls of clay equal in mass ("Make two balls that are the same"). They were then asked to roll one ball into a sausage- or snake-like shape. Not until the age of 8 do most children conclude that the two differently-shaped balls have the same mass.⁶ Lovell concludes:

...the concept of the invariance of weight depends, in part, upon prolonged and varied experience of the physical world. Pupils need to engage in appropriate activities, coupled with discussion with the teacher, to help them come to appreciate that the weight of a body is conserved, regardless of the spatial arrangement of its parts, provided nothing is added or deleted. (p. 170)

Common everyday experiences often hinder as well as support the development of understanding. The common observation that larger objects usually have more mass than smaller objects often leads to an incorrect generalization.⁷ Children in the elementary grades need to learn about the rudiments of density without being exposed necessarily to the term or its definition. They should learn at a very minimum that larger objects do not necessarily possess more mass than smaller objects. I believe that the concept of density, because it requires students to consider two attributes simultaneously (viz., mass and volume), should not be taught formally in the middle years program. It should be delayed until the secondary years and taught in science rather than in mathematics, where its applications can be considered. A comparison of the new science program and a revised elementary and middle years mathematics

program should be done to ensure that the expectations regarding measurement knowledge and skills are reasonably similar and realistic.

The attribute of mass (weight) should be introduced to children by involving them in activities that initially require the lifting and holding of common objects varying in mass. Lovell (p. 168) suggests several useful introductory activities:

1. With eyes closed, the child picks up a number of objects in turn and then puts them in order of mass. He commences with two objects and increases the number until he has four or five.
2. A number of pairs of objects are examined and the pupil writes down which member of each pair he considers is the heavier (more massive) without lifting them. He then picks up each member and checks his result.
3. A pupil mimes a number of scenes: in one he carries a heavy sack on his back, in another he carries a feather.

After these preliminary, informal activities are completed, children should learn to work with scales and balances; in particular, observing and comparing the effects of placing various sized objects on scales and simple balances. As in the learning of other measurement concepts, standard masses and calibrated balances and scales need not be used initially: simple weights include nuts and bolts, nails, clay, stones, dry beans, wooden and plastic blocks, thumbtacks, paper clips, coins; simple balances and scales include pan balances, cups suspended on both ends of a rod, and elastic band 'stretchers.' Once children become familiar with simple measurement devices and the common standard masses of grams and kilograms, they can learn to work with calibrated devices such as kitchen and bathroom scales and more accurate forms of pan balances.

The Understanding of Time

Time is perhaps the most abstract measure to comprehend; its nature discussed and debated by philosophers and scientists. Aristotle (Russell, 1967) posed the supposition that time does not exist, since it is composed of past and future, of which one no longer exists while the other does not yet exist. He rejects this view arguing that time is "motion that admits of numeration" (p. 215). Much later in time (assuming it does exist), Augustine asks: "What then is time. If no one asks of me, I know; if I wish to explain to him who asks, I know not" (p. 352). Augustine is perplexed because neither past nor future, but only the present, really is; the present is only for a moment, and time can only be measured while it is passing. Nevertheless, he admits, there really is time past and future (p. 352). He maintains that three times exist: "a present of things past, a present of things present, and a present of things future. The present of things

past is memory, the present of things present is sight; and the present of things future is expectation" (p. 352). In spite of his attempt to develop a theory of time, he readily admits its inadequacy: "I confess to Thee, O Lord, that I am as yet ignorant what time is" (p. 353).

Elementary school children usually have only a meagre understanding of the passage of time and almost no conception of historical time (Inskeep, p. 69). Lovell reports that the time perspective in children less than nine years of age is very shallow: for example, children entering school usually do not understand that a specified day of the week in a town is the same day of the week in a province or a country. It is not uncommon for primary children to believe that their grandparents witnessed impossibly past events - for example, the era of the dinosaurs.⁸

In the elementary school, teachers should regularly use the common descriptors of time such as the following: morning, afternoon, day, night, month, year, later, earlier, and so forth. Prior to lessons on clock reading, students should become aware that the positions of the hands on a clockface change constantly throughout the day and likewise so do the numbers on a digital clock. To emphasise these changes in position and number, events happening during the day can be labelled with sketches of the clockface or the digital display: for example, moving the hands of a clockface to correspond to a picture of a child emerging from bed, walking to school, eating lunch, and so forth.

Although clock reading is a useful life skill, it likely contributes little to children's understanding of time as it entails largely dial reading. Almost 90% of grade 4 students in the B.C. Assessment were able to interpret a clock face showing "twenty minutes past eleven" with the digital reading 11:20. This impressive result suggests that most work related to time in the elementary school is devoted to "telling time." Clock reading should be continued to be taught because it has the potential to help children apply many different mathematical skills: counting by 5s, simple fractions, for example. Children should also learn the common equivalences such as 60 minutes = 1 hour, 60 seconds = 1 minute, 7 days = 1 week, and so forth.

Children can acquire a better sense of time when daily events are recorded on a calendar. Salient events such as birthdays, class field trips, and holidays are most likely to be recalled by children and, therefore, act as a referent throughout the school year.

Children's sense of longer time spans and for events occurring well before they were born develops slowly. To develop a sense of long periods of time (centuries, for example) and historical events (e.g., World War II), children need much preparatory work. Timelines of personal events can be used to help children relate to historical

events: understanding that their grandfathers were likely children during the Second World War while their great-grandfathers were children during the First World War.

Children often confuse time, space and motion. In a typical Piagetian experiment, two cars are moved from point to point, starting and finishing together at the same point. Young elementary children correctly conclude that the cars start, stop, and "raced" for an equal period of time. However, if the experimenter starts and stops the cars simultaneously but varies their individual speeds as well as their distances travelled, children in the early elementary grades erroneously conclude that the cars travel for differing durations of time. Perhaps their confusion about the pace of an event with its duration is not surprising because many common expressions such as "That was a long lecture" demonstrate similar confusion.

How can elementary school teachers help children develop an understanding of temporal terms and relationships? Perhaps, the best advice is to involve children in events having clear beginnings and endings and to measure such events with various types of simple "clocks" such as the following: ball bounces, water drips, sand clocks, candle burning, heartbeats, pendulum swings, bouncing a weight on spring, flashing lights, metronome, musical notes, handclaps, and so forth.

Children need to learn to order the duration of events. For example, children can time a variety of events using a simple clock and then discuss how to order the length of each event. Opportunities for ordering short events and long events should be provided. For example, children should learn to place pictures of events such as school opening, Halloween, Thanksgiving, and so forth in the order they occurred.

Many children do not understand that the units chosen to measure the duration of an event are arbitrarily chosen, as is the case of all measurement units. Lovell states that "It will be high school before 75 percent of the age group will correctly answer the question 'In springtime, when we advance the clocks by one hour, do we get one hour older?'" Obviously, children need to work with a variety of clocks and units of time before they can be expected to acquire a sound understanding of conventional units of time.

The Understanding of Temperature

As a measure of the amount of heat contained in a body, temperature is sensed easily by children and adults, especially by those living in Saskatchewan.⁹ Before more precise instruments such as thermometers are studied, children should engage in activities such as placing their hands in varying temperatures of water to heighten their sense of temperature and to help them become acquainted with the associated thermal terminology.

Once the students become fluent with such comparative terms as "hotter," "colder," "warmer" and so forth, they should learn how to read and use a standard Celsius thermometer.¹⁰ They should be provided with opportunities to estimate temperatures: "How cold is it today?"; "How hot do you think this water is?"

By the end of the elementary grades, students should learn to associate some temperatures with common events: freezing and boiling temperature of water, normal body temperature, typical room temperature. They should learn how to estimate temperatures using these benchmark data: e.g., estimating that a fluid heated to a temperature of 150 degrees Celsius would be dangerous to handle because its boiling point is greater than that of water. The B.C. Assessment found that only 61% of grade 7 students knew that 100 degrees Celsius is the boiling point of water given the choices 212, 21, and 10 degrees Celsius.

The Understanding of Angles

In the elementary grades, only informal work with angles should be attempted. Children should learn to perceive angles initially as a change of direction or the amount of a turn. For example, children can be asked to draw a straight line and then be asked to change its direction. Children should learn to identify examples of angles seen throughout the classroom: corners of the wall, open blades of scissors, hands of a clock, and so forth. One device I've used is two interlocking paper plates: one white and one red. After making a cut from the edge to the center, the plates are inserted into each other at the cut line, allowing the plates to rotate. The edges can be labelled with degrees to any desired degree of precision. This device is also useful for representing fractions.

Once children can perceive angles, they should learn a variety of ways to compare them. They should be taught to compare two or more angles by superimposing the angles one upon another: paper cutouts of angles are ideal. They should also learn to compare angles by referring to a standard angle measure, especially the ubiquitous right angle or corner angle: "This angle is larger than a right angle because it can contain the corner of a square" or "This angle is smaller than a right angle because it doesn't turn as much as my forearm moves from my shoulder." By folding a paper square children can be introduced to a half right angle and use this angle to acquire more precise measurements.

Teachers should ensure that angle comparisons do not always involve a stereotypic angle orientation (one horizontal arm) or equal length arms. Children should learn to compare and estimate angles seen in a variety of orientations and with different sized arms.

The standard protractor and the standard unit of angle, the degree, should be introduced in the middle grades. Before using the standard 180 degree protractor, which most students find difficult to use and read, they should construct and use their own using simple "wedges," a transparent plastic circle subdivided into a variety of equal sectors or marked off in degrees, or the interlocking paper plates previously described. Likewise, other than the term right angle, there seems little need in the elementary grades to burden children with such technical terms as acute, obtuse, reflex and so forth or the naming of angles in terms of their endpoints. The aim of teaching angles in the elementary grades should be to lay the foundation for the use of the protractor in the middle years.

Some final words: I believe a major weakness in our instruction in measurement is the lack of anything worthwhile to measure. When students are eventually (unfortunately sometimes never) given a measurement device, they measure merely for the sake of measuring. I wonder how many desks and textbooks in Saskatchewan classrooms have been measured, for example? Students are rarely asked to measure for a purpose. Consider as an alternative how much activity and learning, both mathematical and non-mathematical, the following types of open-ended questions are likely to generate:

1. Which colour of human hair is strongest? Which gender?
2. How long would it take a handshake to cross from Regina to Saskatoon?
3. How slow are molasses anyway?
4. How thick is paper?
5. How heavy and how big is 1 000 000 one dollar bills?
6. How fast is an eyeblink? How many times in a day do we blink?
7. How much garbage do our families throw out?

We need to provide teachers with some worthwhile measuring activities where data is gathered, interpreted, and used to solve a problem, examine a situation, or satisfy some curiosity.

Footnotes

- ¹ In mathematics, a transitive relationship R ("related to") can be written symbolically: if $a R b$ and $b R c$ then $a R c$. This principle allows us, for example, to determine the relative size of two lengths "a" and "c" without directly comparing them. Instead, they need to be compared only to a third referent length "b" (if $a R b$, and $b R c$, then $a R c$). Transitivity also permits us to order quantitative information without measuring: for example, ordering the measurements 100 m, 43.3 m, and 345 m.
- ² I frequently ask prospective teachers to calculate the area of a non-right angled triangle by giving them only the dimensions of each side rather than the dimensions of the base and altitude. Except for a very small minority who ask me for the missing altitude or who attempt to use some advanced procedure such as Herod's formula (none have used this formula to date), the vast majority of students simply multiply various pairs of numbers and divide by 2.
- ³ Capacity refers to the internal volume of a container. Common usage makes a distinction between volume and capacity: volume is usually reserved for dry measures, capacity for liquid measures. It is not incorrect to refer to 250 cubic centimetres of water, however; the distinction between this measurement and 250 mL is a matter of custom only.
- ⁴ Many adolescents and adults mistakenly believe that the amount of water displaced by a submerged object depends upon the weight of the object rather than its volume. Children and adults need to acquire many physical experiences in order to eliminate these common misconceptions.
- ⁵ Strictly speaking, in lifting an object, we sense its weight, the earth's gravitational force exerted on that object at that position in space rather than the mass, the amount of substance in a body. Mass is typically measured indirectly by using an equal-arm balance, an instrument that determines when the weights of two bodies are equal, and hence when their masses are equal. The SI units for mass (kilogram) and weight (newtons) are frequently misused in everyday conversation. Expressions such as "I weigh 60 kilograms" are heard regularly. Teachers should be aware of the distinction between mass and weight but should not be concerned when children and other adults use the term weight rather than the correct term, mass.
- ⁶ I have observed some children who conclude that the "snake is heavier because it's longer" while others conclude the "ball is heavier because it's fatter."
- ⁷ The familiar riddle "Which weighs more, a ton of feathers or a ton of steel?" typically fools most children and some adults who have a limited understanding of density.
- ⁸ Lovell reports two early studies on children's understanding of time in which pupils were asked if their parents and grandparents were alive in 1492 when Columbus set out for the New World. Both investigations demonstrated that this question was not understood by 75 percent of pupils until the age of nine.

- ⁹ Temperature is a more complex measure than is commonly believed. Scientific usage considers temperature as a property that determines whether or not a system is in thermal equilibrium with other systems. When two or more systems are in thermal equilibrium, they are said to have the same temperature.
- ¹⁰ Non-standard thermometers other than body sense are difficult to create. Some suppliers do have unmarked thermometers that allow children to experiment with gradations of their own choice before they are introduced to standard scales. Teachers can create their own non-standard thermometers simply by placing a strip of masking tape alongside a regular thermometer and allowing children to design and experiment with a variety of scales.

CHAPTER 12

GEOMETRY AND SPATIAL SENSE

Despite the historical significance of geometry, basic questions continue to be asked about its nature and its role in a mathematics program and, judging from the results of several surveys (see the NACOME report, 1975, for example) teachers in the elementary and middle years present few geometrical ideas to children.¹ The Second International Mathematics Study [IMS] (Robitaille, O'Shea & Dirks, 1982) reported that British Columbia grade 8 teachers assigned less importance to the study of geometry than all other participating countries (p. 21).

Perhaps, the lack of systematic instruction is one reason that students do not perform well on tasks requiring a knowledge of geometry. For example, the 1982 National Assessment of Educational Progress (NAEP) found that fewer than 10% of 13-year-olds could find the measure of the third angle of a triangle given the measures of the other two angles. Over eighty percent of grade 8 teachers in British Columbia claimed they taught the Pythagorean theorem, yet only 42% of the grade 8 students could answer correctly an item that required them to calculate the length of the hypotenuse of a right triangle pictured with dimensions of 3 and 4 units (Robitaille, O'Shea & Dirks, 1982, p. 221).

Usiskin (1987) found that senior high school students possessed a sparse knowledge of geometry when they entered a geometry course. Only 63% correctly identified two triangles shown in a diagram of four polygons. The remainder of students identified only the triangle approximating an equilateral triangle and ignored the obtuse triangle also pictured in the diagram. The tendency of some students to classify figures using a visual prototype, "an approximately perfect triangle," will be explored further in this chapter.

Although not a goal of the elementary and middle years program, the study of Euclidean geometry in high school is claimed to help students understand and construct deductive proofs. Usiskin (1987) reports that, at the end of a geometry course in which "proof-making" was studied long enough for teachers to consider a test on proof-making to be fair, 28% of the students could not do the easiest triangle congruence proof, and only 31% were judged to be able to understand the proof-making process (p. 19). Not surprisingly, performance on the proof-making test administered at the end of the year was reported to be highly correlated with students' knowledge of simple geometry facts at the beginning of the year (p. 19).

The Goals of Geometry Instruction

Teachers need to understand the purpose of studying geometry if its study is to be included in all grades in the K-12 mathematics program. After examining the mathematics education literature, O'Daffer (1980, p. 94) identified several reasons commonly given to support the teaching of geometry:

1. To provide for the child's everyday needs.
2. To provide for the child's developmental needs.
3. To achieve subject matter and content goals and to develop a cultural awareness.
4. To achieve process goals.

Everyday Needs

Whether geometry meets the everyday needs of children depends entirely on the type of instruction provided and the out-of-school situation in which children live. In everyday life, people are surrounded by planar shapes and solids and occasionally have a need to refer to their geometric names. Common language contains many geometric terms (intersection, perpendicular, angle, and so forth). Other than these trivial applications, however, people rarely apply the geometric terminology, concepts, principles and procedures taught in school.

Since elementary and middle years teachers tend to emphasize what they see as "practical," they often ignore instruction on topics that may not be seen as directly practical but whose study may ultimately contribute to the understanding of other mathematical ideas. For example, few elementary and middle years teachers challenge students with visualization problems. In the 1985 B.C. Mathematics Assessment (Robitaille & O'Shea), students were presented with a schematic of a cube with one edge drawn with a thick line. They were asked to determine the total number of edges. About half of the grade 4 students and about 70% of the grade 7 students chose the correct answer, 12 edges. About 20% of the grade 4 students chose 9 edges, the number of exposed edges of the cube, as the answer. Interestingly, about the same number of grade 7 students made the same error of not visualizing the hidden edges.

Developmental Needs

Many developmental psychologists (Piaget, Flavell, Dienes, for example) argue that children's action on objects influences their learning of mathematical ideas. O'Daffer claims that "since objects, even those especially prepared for developing concepts in elementary school mathematics, such as Cuisenaire rods, Dienes multibase blocks, logical blocks, and the like, are essentially geometric in nature, the developmental theories would seem to support the need for a geometric component in the cur-

riculum" (p. 95). However, this view seems too broad, implying that almost any topic can be considered as part of geometry.

Many mathematical tasks clearly have a geometric component: reading and interpreting graphs, measuring, and even writing, sequencing, and placing numbers correctly (p. 97). Other mathematical tasks can be developed and understood from a geometric perspective: learning to operate with and compare numbers positioned on a number line, for example. Consequently, a knowledge of geometry will support the learning of other mathematical ideas, and, as Usiskin argues, will prepare students for the more advanced ideas presented in high school geometry courses.

Many theorists have identified, what is loosely referred to as, spatial abilities. These abilities are presumed to underlay intellectual development. For example, Bishop (1983, p. 182) describes two types of spatial ability: (1) the ability to mentally manipulate, rotate, twist or invert a pictorially presented stimulus object; (2) the comprehension of the arrangement of elements within a visual stimulus pattern and the aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented.

The Soviet psychologist, Krutetskii (1976), in discussing individual differences in preferred modes of processing, distinguishes between people he terms "analytic" and those he terms "geometric." Analytic reasoners can be characterized by:

... an obvious predominance of a very well developed verbal-logical component over a weak visual pictorial one. They operate easily with abstract schemes; they have no need for visual supports for visualizing objects or patterns in problem-solving, even when the mathematical relations given in the problem "suggest" visual concepts. (p. 317)

Geometric reasoners, however, exhibit thinking characterized by:

... a very well developed visual-pictorial component ... These pupils feel a need to interpret visually an expression of an abstract mathematical relationship and demonstrate great ingenuity in this regard: In this sense, relatively speaking, figurativeness often replaces logic for them. (p. 321)

Researchers have investigated the apparent differences in spatial abilities between boys and girls (see, for example, Fennema & Tarter, 1985). Although the methodology and, consequently, the findings often vary from study to study, boys typically outperform girls on this intellectual dimension. Whether these differences, if they exist, warrant special attention in program revision is debatable.

Subject-matter Goals

Because geometry is a branch of mathematics, students cannot be considered mathematically literate unless they understand at least the basic geometric ideas. Thus, geometry should be studied for its own sake. In the elementary and middle years program, students should learn at the very least to categorize planar shapes and solids and investigate their properties.

An important feature of geometry is its relationship to other subjects. For example, students should be exposed to notions of symmetry, translations, and tessellations through a study of art forms and learn to apply these notions to their environment.² Students should learn to construct designs using a variety of instruments and techniques (compass, straight-edge, template) and build constructions of solid shapes using crude blueprints.³

Teachers will find that geometry, when treated as a practical, "hands-on" subject, is unusually effective in arousing children's curiosity and promoting their active involvement. Geometry needs to be more than the dry, dull, pedantic stuff typically presented in most textbooks.

Meeting Process Goals

As Saskatchewan teachers try to incorporate the Common Essential Learnings (CELs) into all aspects of their teaching, geometry, if taught properly, can stimulate thinking and involve children in the processes commonly used to acquire knowledge in any subject area. These processes include analysing, conjecturing, generalizing, and verifying. This list is certainly not exhaustive and likely many other processes can be included as well.

Analysing involves the examination of parts and relationships. Through the process of analysis, various components, attributes, claims, assumptions and reasons are identified and distinguished. Analysis can be analysed itself and separated into two components: recognizing attributes and components and recognizing relationships and patterns.

This first component of analysis requires an individual to recognize and then articulate the parts that together constitute a whole. Attempting to recognize attributes should help students focus on the details and the structure of objects, ideas, designs, and so on (ASCD, 1988). To develop this aspect of analysis in mathematics, children should begin analysing commonly seen objects for their mathematical properties or attributes. In the elementary grades, children should be challenged to identify as many properties as possible of plane figures, geometric arrangements, and other concrete representations. For example, in analysing a hexagon, children might identify the number of sides, the number of diagonals, the measure of the interior and exterior angles, and so on. Questions such as the following should be posed regularly: "Can

you draw a triangle having two right angles? Why or why not?" "What can you tell me about isosceles triangles?"

In analysing geometric properties, students should learn to recognize the variant and invariant properties: for example, learning that hexagonal-shaped objects can be any colour, texture, thickness, size and weight, and vary in the measures of their internal angles and the lengths of their sides. In order to identify the attributes that children consider are important, researchers have presented open-ended tasks such as the following: "Draw several four-sided figures that are different in some way and explain why they are different." I have found through informal assessments that most children in the elementary grades restrict their drawings to rectangles varying in size but with horizontal and vertical sides. Few children draw asymmetrical quadrilaterals, for example. Furthermore, I have seen grade six students who still think that a right-angled triangular cutout loses its "right-angledness" when the triangle is "tipped over," in other words, the base is moved to a non-horizontal position.

Teachers need to be aware that students who use the correct geometric terminology are not necessarily familiar with their meanings. Hoffer's research (1983) indicates that students typically have gross misconceptions or totally incorrect ideas about the attributes of common geometric shapes. He provides the following examples of misconceptions:

Many students process "triangle" to mean an equilateral triangle with a side parallel to a line parallel to the student's eyes. Students actually will turn the page or physical object to view the triangle in what we might call "the textbook position." Other students, through junior high school, allow figures to be called "triangles" even when the sides are curved outward or inward. They seem to focus on figures with "three points" and not three "straight" sides. Figures whose sides curve outward are more readily called triangles as long as they do not depart too far from equilateral. Many students, through junior high school, hesitate to (or definitely do not) allow scalene triangles to be called "triangles," especially ones in which a side length is considerably greater than the corresponding altitude, or vice versa. Students consider these to be "lopsided" or "too narrow" to be triangles. There are similar misconceptions about quadrilaterals. (p. 215)

Hoffer also reports that students at all ages use physical properties to describe figures. They will state that equilateral triangles are "perfect triangles," while other triangles are labelled "too tall," "flat," "looks like an arrowhead," and so forth.

Conversely, students can often demonstrate good understanding of geometric concepts but exhibit difficulty in describing these concepts using standard mathematical

terminology. For example, students commonly refer to parallel lines as "both are straight," and perpendicular lines as "going straight up" (p. 215).⁴

Teachers should not be afraid to use the correct terminology when discussing geometric concepts, or any mathematical topic for that matter. If young children can use the phrase "Teenage Ninja Mutant Turtles," then they certainly can learn to use correct terminology. But the correct terminology should be learned while being engaged in more productive activities than are typically seen in school, the "Today we are going to learn about rectangles" type of lesson. Teachers might have students construct a geometric pattern, for example, and encourage students to use the correct terminology to describe their design: "I used 5 squares and 4 rectangles to make a house."

Teachers should also show how knowledge of common geometric terms and their root words can be used to analyze other words: for example, using "acute" to describe people as well as angles and triangles, relating the prefix 'poly' to words such as "polygamy" or "polyglot." Even the mouthful "truncated stellated icosahedron" can be deciphered if one has the following knowledge: truncated (to cut or amputate), stellar (star-like), icoxa (20), and hedron (seat, base or face).

Research on concept attainment suggests that directing students' attention to important attributes of examples and non-examples promotes learning of well-structured concepts. Teachers need to spend more time helping students recognize the properties of quadrilaterals, for example, and how they differ from other types of planar objects. A good rule of thumb for teachers to follow is that **if students cannot provide examples and non-examples for mathematical ideas, and express mathematical ideas in their own words, then they do not have an adequate understanding.**

As children mature mathematically, they should be encouraged to examine groups of materials for their common and non-common attributes: for example, determining how triangular cutouts are alike and unlike. Recognizing commonalities is a precursor of classification.

When students identify attributes and components, they make distinctions among elements that constitute a whole; when identifying patterns and relationships, they articulate the interrelationships among these components. The relationships may be causal, hierarchical, temporal, spatial, correlational, or metaphorical (ASCD, p. 91).

Obviously the ability to identify patterns and relationships depends heavily on content knowledge - a main reason why reasoning cannot be taught in isolation from a subject.⁵ To help students develop an ability to recognize geometric patterns (or any other pattern for that matter), we need to provide them with the underlying

background knowledge and, of equal importance, with opportunities to analyse patterns and relationships, which few teachers currently do.

Classifying is the process of grouping items into categories on the basis of their attributes. Much mathematical thought involves classifying. Numbers are classified as prime or composite, even or odd, rational or irrational, and so on. Classification is not done arbitrarily; it is done to help organize information into manageable units. In this way, new information can be linked to known information.

When classifying objects or ideas, common attributes must be identified. Thus the ability to classify depends largely on our recognition of attributes or properties: the more attributes and properties known, the more likely one will be able to classify objects. As has been discussed previously, classifying as a critical thinking skill is not generic: that is, it is difficult to imagine how learning to classify in one discipline (music, for example) can help one classify in another discipline (mathematics, for example) when the knowledge bases are so different. Once an attribute is recognized, the remaining task in classifying is to examine other objects or ideas to determine if they have this attribute in common.

Younger children should begin by classifying concrete objects and be encouraged to explain their classification criteria: "This pile includes triangles, and that pile contains squares." The attribute blocks, colour cubes, and Creature Cards developed originally for the Elementary School Sciences (ESS) project and later published by McGraw-Hall are very useful for helping children learn to classify.

Typically, in "doing mathematics" (as opposed to learning mathematics), a pattern or relationship is recognized in a situation, and a generalization - a proposition that is true over a limited domain - often follows this recognition. **Generalizing or conjecturing** should be considered an important outcome of geometric study, even in the elementary grades. Currently, mathematics programs provide students with little opportunity to generalize. Typically, generalizations are given rather than developed and the task remaining for the student is to provide a verification where the proposition is invariably true.

Teachers can help children make generalizations by providing instruction that regularly asks them to make predictions. I have observed even pre-schoolers who can predict correctly the results of transformations made on concrete objects. One task I gave to two 4-year-olds illustrates this point. I showed them a solid object (either a sphere, cone, cube, cuboid, or square pyramid) and asked them to guess what shape would be displayed on the screen when the object was positioned in a particular way on an overhead projector. (The children had an opportunity to experiment with various objects beforehand: placing hands, pencils, coins, and so forth on the projector.) Initially, both predicted a "ball" would appear on the screen when a ball was

placed on the projector. After seeing a circle appear on the screen, both students correctly predicted the shape of a circle when the task was repeated later using different sized balls. Surprisingly, both students predicted a triangular shape for a cone placed on its "side" and a circular shape for a cone placed on its base.

Teachers can promote good thinking by regularly asking "what would happen if" questions: "What would happen to the shape of the clown's nose (cone made from molding clay) if we cut off its tip?;" "What shape would we get if we fit these two congruent right angles together along their longest sides?;" "What would happen to the perimeter if we placed these two rectangles together?;" "What would happen to the area of a square if we double the length of each side?"

Verifying propositions has traditionally been the mainstay of high school geometry. However, verification has been rigidly restricted to deductive proofs arranged in the standard two-column format. Beginning in the elementary grades, children should learn other ways to reasonably "verify" propositions. For example, children should be taught to evaluate propositions simply by enumerating a few cases. To test the validity of the proposition that "The diagonals of a quadrilateral bisect each other," children should be encouraged to draw and measure a variety of different quadrilaterals. They should also be encouraged to restate propositions that are true for a limited range of values. Even when it is impossible to enumerate for all cases, a few well chosen cases can either disprove a proposition or lead to new insights about geometry.

Children should also be encouraged to create and test new propositions by modifying some condition of the original proposition as illustrated in the two examples below:

1. If the diagonals of a square bisect each other, is it true that the diagonals of a cube intersect each other?
2. If squares form a tiling pattern (tessellate the plane), will circles also?

Developing a Framework for Instruction in Geometry

Geometry instruction often suffers from a lack of consensus about its goals, the expected learning outcomes, and the nature of its instruction. What is needed is a framework based on how students learn geometry and acquire a spatial sense. Pierre van Hiele and Dina van Hiele-Geldof (Hoffer, 1983; O'Daffer, 1980; Burger & Shaughnessy, 1986) have developed a framework based on children's geometric reasoning that has been used extensively to guide further research into children's learning. Their framework holds promise for evaluating the geometry curriculum, and establishing suitable goals and selecting appropriate instructional activities.

The van-Hiele thought levels as they apply to geometry are summarized in Table 1. The first three levels 0, 1, and possibly 2 describe the development of thinking of students in the elementary and middle grades. For the sake of completeness, I have included a description of Levels 3 and 4, which are more suited to an analysis of thinking of secondary and university students. The Soviets have used the van-Hiele framework to analyze student materials and to carry out related research into student learning. Hoffer reports the following findings pertaining to geometry instruction in the elementary and middle years:

1. In the first five grades, a significant number of children perceive figures as wholes. For example, they experience difficulty in recognizing imbedded figures.
2. Pupils stayed at the first level of geometric development for a considerable time. By the end of grade 5, only 10% - 15% reached level 1, which is needed for further study of geometry. With respect to pupil understanding of solids, the delay was even longer, that is, until grade 7.
3. Pupil's development was altered by purposeful study of geometric material. Familiarizing second graders with solids enabled them to reach level 1, surpassing the progress of seventh graders in the traditional school. (p. 209-210)

The importance of systematic instruction appropriate to a child's level is underscored by Pyshkalo: "In experiments in recent years it has been shown that the capacity in children of primary school age for ordered thinking is significantly higher than has usually been believed and higher than the level at which traditional instruction begins" (Hoffer, p. 210).

Burger and Shaughnessy (1986) investigated the van Hiele's levels of development of students from grade 1 through 12 on tasks such as drawing shapes, identifying and defining shapes, sorting shapes, determining a mystery shape, establishing properties of parallelograms, and comparing components of a mathematical system. They found that the behaviour of students could be analyzed using the van Hiele's levels, although the discreteness of the levels, particularly analysis and abstraction was not confirmed. Table 2 contains summaries of their findings applicable to children in the elementary and middle grades.

In considering the findings of studies of van Hiele's thought levels, teachers need to understand that concept formation in geometry appears to develop slowly over many years and only when systematic instruction is provided. Unless students acquire a good grounding in geometry and develop a sound spatial sense in the elementary and middle years, the study of geometry as a formal, deductive system will be meaningless, forcing students to memorize definitions, principles and theorems to be regurgitated during exam time and subsequently forgotten soon after.

Table 1. The van Hiele's Thought Levels related to an Understanding of Geometry and Space. (Based on the analysis by Hoffer)

Level 0 Visualization. Students recognize figures by their global appearance. They can say "triangle," "square," "cube," and so forth, but they do not explicitly identify properties of figures (e.g., not recognizing that rectangles have right angles or that opposite sides are parallel). They can also distinguish between these figures. Students at this level can reproduce figures by drawing or using a geoboard.

Level 1 Analysis. Students analyze properties of figures: for example, rectangles have equal diagonals; triangles have 3 sides; isosceles triangles have two congruent sides. However, they cannot explicitly interrelate figures or properties. That is, they might not realize that isosceles triangles contain two congruent angles as well as sides primarily because they consider only single attributes. At this stage, definitions are not well understood. Students also do not understand the existence of nesting: i.e., that figures have more than one name (a square can be termed a rectangle, a parallelogram, a quadrilateral, for example)

Level 2 Abstraction. Students relate figures and their properties: for example, knowing that every square is a rectangle. But they cannot justify their observations with orderly sequences of relevant statements. They can, however, logically order figures and understand that definitions provide a complete description of the necessary and sufficient conditions or attributes for classifying figures.

Level 3 Deduction. Students develop sequences of statements to deduce one statement from another, such as showing how the parallel postulate implies that the angle sum of a triangle is equal to 180 degrees. (The major goal of the study of Euclidean geometry in secondary school.) However, they do not recognize the need for rigor nor do they understand relationships between other deductive systems.

Level 4 Rigor. Students analyze various deductive systems with a high degree of rigor comparable to Hilbert's approach to the foundations of geometry. They understand such properties of a deductive system as consistency, independence, and completeness of the postulates. They understand that many geometries exist and can be investigated in the absence of concrete models. (A goal likely intended for university mathematics majors or very gifted secondary students.)

Table 2. Characteristics of Children Classified at Van Hiele's Level 0 and 1. Level 0

-
1. Use of imprecise properties (qualities) to compare drawings and to identify, characterize, describe, and sort shapes.
 2. References to visual prototypes to characterize shapes: for example, a rectangle looks like a door.
 3. Inclusion of irrelevant attributes when identifying and describing shapes, such as the orientation of the figure on the page.
 4. Inability to conceive of an infinite variety of types of shapes: for example, one fifth grader postulated that about 12 different triangles can be drawn, but later revised his estimate to over 1000 triangles.
 5. Inconsistent sortings; that is, sortings by properties not shared by the sorted shapes or not including all shapes in a sorting: for example, third and fifth graders tended to sort triangles according to how closely they approximated an equilateral triangle.
 6. Inability to use properties as necessary conditions to determine a shape: for example, guessing the shape in the mystery shape task after far too few clues, as if the clues triggered a visual image.
-

Level 1

1. Comparing shapes explicitly by means of properties of their components.
 2. Prohibiting class inclusions among general types, such as quadrilaterals: that is, not considering squares as rectangles.
 3. Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry, and so forth.
 4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape.
 5. Descriptions of types of shapes by explicit use of their properties rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles.
 6. Explicit rejection of textbook definition of shapes in favour of personal characterizations.
 7. Using empirical methods to test the validity of a proposition: for example, relying on a variety of drawings and making observations about them.
 8. Explicit lack of understanding of mathematical proof.
-

Planning a Geometry Curriculum

In order to improve the quality of instruction in geometry and to capture the interest of elementary and middle years students, teachers need to be provided with instructional ideas different from the pedestrian ones traditionally presented in textbooks.⁶ As a first step, the curriculum should be planned and organized according to some established scheme. I believe that such a scheme can be developed by combining the research findings generated by the van Hieles and by researchers investigating spatial perception.

Applying van Hiele Based Research

The van Hiele research can be used to develop a geometry program that is developmentally sound yet practical for teachers to use. Teachers in the elementary grades in which the majority of children will display the characteristics of the first two van Hiele levels should provide basic exploratory experiences using cutouts, paper folding, pattern blocks, geoblocks (construction blocks), molding clay models of solids, geoboards, D-stix (coloured plastic rods that fit into multi-pronged vinyl connectors), straws, grid work, templates, tessellations, tangrams, mirror cards, maps, and so forth. Crowley (1987) identifies several examples of activities appropriate for the first four van Hiele levels. The discussion that follows is based on her thoughts on the first two levels.

At Level 0 (Visualization), children recognize geometric shapes by their global appearance without referring to their physical properties. Students at this stage should be provided with the following types of instructional activities:

1. Manipulating, colouring, folding, and constructing geometric shapes: e.g., using pattern blocks (rectangles and squares) to make a pleasing design.
2. Identifying shapes or geometric relations in simple drawings, in a set of cutouts or pattern blocks, in a variety of orientations (especially not the stereotypical textbook orientations),⁷ in the environment (shapes and solids naturally seen in the home, school, photographs, drawings, etc.), and imbedded within other shapes (e.g., find the parallel lines in a trapezoid).
3. Creating shapes by copying figures on dot paper, grid paper, or tracing paper, by using rectangular and circular geoboards, or by tracing cutouts or using templates, by drawing figures freehand, by constructing figures with sticks, straws, or pipe cleaners or by tiling with pattern blocks, and so on.
4. Describing geometric shapes verbally using appropriate standard and nonstandard language such as the following: cube (block, box, dice shape), sphere (ball), cone (ice cream cone), angle (corners, turn), cylinder (can), and so forth.

5. Solving problems by measuring, and counting shapes: e.g., finding the area of a box top or the volume of a box by tiling and counting; fitting triangular shapes to produce a rectangle (tangrams); completing 'jigsaw' puzzles with standard and nonstandard figures.

At **Level 1 (Analysis)**, students begin to recognize properties and the focus of instruction changes accordingly. Students at this stage should be involved in the following types of instructional activities:

1. Measuring, colouring, folding, cutting, modelling, and tiling with the aim of identifying properties of figures and other geometric relationships such as drawing a kite-shaped figure on grid paper and then folding its cutout to determine the fit.⁸
2. Describing a class of figures by its properties (through charts, verbal statements, and 'property cards'): for example, "take the picture card and describe its features so that your partner can draw it without seeing the picture or hearing its name;" "take these property cards and place them on the proper shapes."
3. Comparing shapes according to their characterizing properties: noting how squares and rhombuses are alike but different in regards to angles, and sides, for example.
4. Sorting shapes by single attributes: e.g., sorting cutouts of quadrilaterals by number of parallel sides, number of right angles, and so forth.
5. Identifying and drawing a figure given an oral or written description of its properties: e.g., describing a figure verbally and asking for all possible figures possessing those properties; providing students with successive clues ("What's my name?" or 'Who am I?') until the figure is identified.
6. Identifying shapes from visual clues by gradually revealing a shape (slowly uncovering a masked figure displayed on an overhead or sliding a cutout out of an envelope) and asking students to identify at each stage possible names for the shape. An alternative to visual clues is to provide a tactile stimulus such as 'feeling' objects placed in a cloth bag, tracing outlines of objects while blindfolded, or tracing figures on another child's back.
7. Deriving empirically standard rules, formulas, and generalizations such as the formula for the area of a rectangle by tiling and measuring many types of rectangles.
8. Characterizing and contrasting the properties of different classes of figures: "The statement 'opposite sides are equal in length' describes which figure?"; Exploring the relationship between the diagonals of quadrilaterals formed by the fastening of two

cardboard strips at various lengths ("What can you say about the diagonals of a square?", "Do other quadrilaterals share this property?").

9. Discovering properties of unfamiliar classes of objects: from examples and non-examples of trapezoids, determine the properties of trapezoids (similar to the activities presented in 'Creature Cards').

10. Encountering and using appropriate vocabulary and symbols.⁹

11. Solving problems requiring a knowledge of properties of figures and geometric relationships: for example, without measuring, find the sum of the angles in a rectangle or in any regular hexagon.

Applying Spatial Perception Research

Many activities can be derived from the tasks commonly used by researchers to investigate spatial perception - the ability to recognize and discriminate stimuli in space and to interpret those stimuli by associating them with previous experiences (Del Grande, 1987, p. 126). Researchers have identified the following abilities that seem to have the greatest relevance to academic development and to curriculum development: (1) eye-motor coordination; (2) figure-ground perception; (3) perceptual constancy; (4) perception of spatial relationships; (5) visual discrimination; and, (6) visual memory. None of these abilities pertain exclusively to performance in geometry; instead, all school subjects involve these general abilities.

Three points must be made before these abilities are discussed. First, unless a child is extremely handicapped, most abilities can be acquired through program intervention, and for many children they are acquired incidentally through involvement in other academic experiences (reading charts and graphs, for example). Second, instruction is rarely limited to the development of only one ability. Most activities help develop several spatial perception abilities. Third, in the main, most of the suggested activities should not be done for their own sake. Instead, they should be incorporated into a more meaningful activity intended to teach some geometrical or mathematical notion. For example, learning to use a compass and straightedge to construct figures can be used to improve children's hand-eye coordination, while providing new insights into the properties of geometric figures. In selecting an activity, teachers should ask themselves this question: What mathematics can be incorporated into this activity?

Eye-motor coordination requires the coordination of vision with body movement, particularly in carrying out manual tasks. Children who experience difficulties in this area have trouble joining dots on geoboard paper, cutting out and tracing figures, drawing freehand sketches, using rulers, stacking cubes or arranging tiles, and other such tasks. In extreme cases, children who have problems with eye-motor coordination attend more to the manual task than to the intended mathematical task.¹⁰

Activities such as the following can be beneficial: Copying prototype figures with and without guidelines (connecting dots, picture completion, following a frame of reference such as a clockface); Completing pictures and sketches of outline drawings with various degrees of obliteration: for example, presenting a triangle in which all vertices have been erased; Joining complementary objects (jigsaw puzzles, nuts and bolts, hooks and washers, and so forth).

Figure-ground perception requires the visual identification of a figure or object imbedded in a picture and the discrimination of relevant from irrelevant background features. Activities include those involving intersecting lines, intersecting figures, hidden figures, overlapping figures, figure completion, figure assembly, similarities and differences, and the reversal of figure and ground (Del Grande, p. 130). Typically, students are asked to identify intersecting areas by pointing, tracing, or colouring: for example, "Colour the space where the triangle meets the rectangle."

Figure-assembly activities (also requiring hand-eye coordination) involve the filling of a figure using geometric objects such as pattern blocks, tangrams, and felt and cardboard cutouts: for example, completing an outline of a kite using a variety of different sized triangular shapes.

Perceptual constancy - the constancy of shape and size - requires the recognition that an object has invariant properties such as size and shape in spite of the variability of its impression as seen from a different viewpoint (Del Grande, p. 128). For example, children need to understand that the shape and size of a solid object as well as other properties (its number of edges and vertices, for example) remain constant regardless of the changing perspective of the viewer.

Activities that require the student to identify an attribute of an object both before and after a re-orientation are useful: for example, counting the vertices (corners) of different sized cubes placed in different orientations.

Perception of spatial relationships requires an understanding of the positional relationships of objects and the observer. Students need to learn that certain mathematical transformations on figures and objects can produce

mathematically different but congruent objects: for example, understanding that a right angle triangle and its mirror image are congruent but, in this instance, can be considered as being not the "same."

Activities such as the following in which objects are viewed from different orientations in the plane or in space can be used: Matching figures in which shaded or coloured parts have been oriented in different ways or requiring students to hold solid objects in the same orientation as some prototype. These type of activities can help expand

children's knowledge of spatial terminology listed in the chart below (based on Bishop, p. 189):

above	below	right	left	far	near	furthest
nearest	in front	behind	opposite	forward	backward	between
middle	side	turn	inside	outside	bottom	top
over	under	beyond	up	down	north	south
east	west	vertical	horizontal	parallel	slanted	perpendicular

Teachers can readily evaluate children's understanding of positional vocabulary through the use of the following activity. The students are placed in pairs or small groups. One child is given a "location card", a simple diagram of objects placed in some type of referent system. The other children are given only the referent system. This student is expected to use positional vocabulary to help other students identify the location of the object: e.g., "It's in the bottom right corner;" "It's to the right of the fridge, close to the stove."

To make the activity more flexible, actual shapes rather than drawn figures should be used and moved to different locations. Bishop reports that this is an extremely revealing task, showing the extent to which some form of coordinated reference system has been mastered. This activity can be modified to include real maps or sketches: for example, "Find the closest lake south of Saskatoon."

One activity suggested by Lappan and Schram (1989) provides children with opportunities to communicate precise information involving spatial and measurement terminology and relationships. Pairs of students face each other with a barrier between them so that they cannot see what is on a partner's desk - an alternative is to arrange the pairs back to back. Each partner attempts to describe a simple construction in sufficient detail that the other partner can re-create the construction: for example, "Place four equilateral triangles on each edge of the square." Later the students compare their creations and discuss the results. As children become more confident more complex shapes and objects can be incorporated into the activity and, for variation, written directions can be substituted for verbal ones.

The understanding of **spatial relationships** can be strengthened through a study of motion geometry: activities involving slides, flips and turns including identifying the motion or the resulting images, drawing images, and drawing patterns. Such activities are best done on a reference system such as grid paper, geopaper and geoboards to help children orient the objects and their images. For example, children can be shown a right-angled triangle and its image on a geoboard and be instructed to continue the pattern by flipping. The Mira is an inexpensive but useful tool to help students form reflections.

One aspect of this spatial ability includes relating the position of two or more objects, noting similarities and differences, finding the shortest path to a goal, completing a figure, completing a sequence, and assembling parts (Del Grande, p. 132). Activities involving the replication of simple planar or block structures should also be used: "Make a tiling pattern like mine" or "Use your dot paper to copy my figure."

One very important aspect of visual perception is the relationship of solid objects to their conventional planar representations. Many children do not understand the conventions of isometric and oblique drawings used in textbooks and in tests to represent commonly seen geometric solids.¹¹ For example, in the 1977 National Assessment only 7% of 9-year-olds and 24% of 13-year-olds correctly answered a question supposedly indirectly evaluating the understanding of volume but more likely evaluating the understanding of drawing conventions. Students were shown an isometric drawing of a 3 by 3 by 4 rectangular solid and were asked to determine the number of cubes. One of the more popular answers of the 9-year-olds (45%) and the 13-year-olds (26%) was 33, the number of exposed faces. Ten percent of the 13-year-olds answered 66, likely doubling the number of exposed faces to account for the unexposed faces.

Activities that require students to build constructions from a schematic or a 'blueprint' are a good starting point: for example, in working with "Geoblocks," students learn to build simple to complex constructions illustrated pictorially photographs and in line drawings; D-stix can be fitted together or straws and toothpicks can be fastened with molding clay at the corners to form skeletal structures that can be oriented to reveal the hidden edges commonly represented by dotted lines in two-dimensional representations.

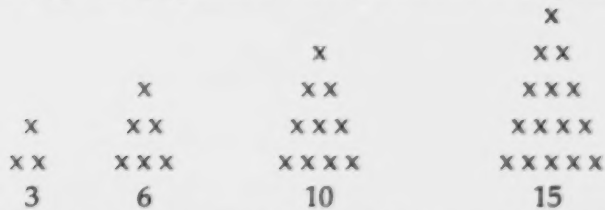
Another activity to help students learn about perspective is to ask students to determine the position of the camera as it took pictures of common objects seen in the classroom. Nets of polyhedra should also be investigated. Nets are planar outlines of polyhedra that when folded and taped form a polyhedron.

Templates of planar shapes can aid children in drawing representations of commonly seen solids. For example, a rectangular prism can be constructed easily with a rectangular template by connecting the vertices of two rectangles drawn in parallel but in different vertical positions. Isometric dot paper should also be used to help children construct diagrams of commonly seen solids.

As children become confident, they should be asked to view constructions from several orientations and draw the construction as it appears to them. This is a difficult task and progress develops slowly. Typically children attempt to represent all surfaces of 3-dimensional objects including the ones normally hidden from view.

Visual discrimination requires that objects are compared and differentiated by any number of visual stimuli. Visual discrimination is essential in conceptual development. The student must be able to discern differences in spatial properties and relationships in order to make generalizations from patterns such as the one below:

Complete the pattern to find the next 'triangular' number:



Visual memory requires that the name, position, or some other geometric attribute of an object be recalled shortly after being hidden from view. The following activity using an overhead projector is one example: "After the projector is turned on, examine carefully the shape and order of all three

shapes you see. When I turn the projector off, draw what you remember." The complexity of the task (the number and types of shapes, the number of attributes, and so forth) can vary according to the maturity and mathematical knowledge of students.

Visual memory plays an important role in activities that require students to imagine objects and to change their imagined orientation:

Close your eyes and think of a cube. Count its corners. How many? How many faces do you "see"? Look on top of it. What shape do you "see"? Imagine a knife slicing off a corner. What shape is the surface of the sliced corner?

Think of a cone. Look down on top of it. What shape do you "see"? Look underneath. What shape do you "see"? Cut off its tip halfway down. What shape do you "see"?

To briefly summarize, in a revised geometry curriculum, children should be actively involved in exploring geometric and spatial relationships. Children must learn that geometry is much more than the rote memorization of terminology and symbolization; instead, geometry must be seen as merely another way of viewing our world.

Footnotes

- ¹ One member of the Mathematics Advisory committee recently quipped, "Geometry is the subject that teachers plan to reach at the end of the school year hoping that the year ends before they reach it." I replied, "Or until the student teacher arrives in May."
- ² I reviewed several art books including Art: Meaning Method and Media (Benefic Press), Discover Art (Davis Publications), and Streetscape (Vision). These books present activities that students would enjoy while learning something about geometry and space: making a repeated pattern, making a model, and sculpturing, to name a few.
- ³ Students should be exposed to some elementary concepts of architecture and drafting. Several years ago, one grade 5 student, who participated in a "mathematics fair" sponsored by me and my students in one of the Education courses I teach, demonstrated the strength of forms built from triangular shapes. As one part of his presentation, he placed several books on top of tetrahedrons and cubes he had constructed out of toothpicks. The tetrahedrons supported considerably more weight than the cubes. Several books I examined contain a number of good ideas for introducing children to the field of architecture: Architecture is Elementary: Visual Thinking through Architectural Concepts (Gibbs M. Smith, Inc.), Beginning Experiences in Architecture: A Guide for the Elementary School Teacher (Van Nostrand Reinhold Co.), The Joy of Building (Van Nostrand Reinhold Co.) and the fascinating Why Buildings Stand Up: The Strength of Architecture (W. W. Norton & Co.).
- ⁴ One 3-year-old I interviewed described a cone as a "Pee-tee. You know those houses that Indians used to live in." Teachers can use children's non-standard language initially, and then use the standard forms interchangeably ("can anyone find the balls or spheres in this box"), and finally encourage children to use only the standard form, when appropriate (golf balls are not referred to as golf spheres, for example).
- ⁵ Studies of chess players, for instance, demonstrate that experts are far better than non-experts in reproducing the position of chess pieces after viewing them in midgame position for 5 to 10 seconds. The experts can correctly reproduce the positions of about 20 to 25 pieces, while ordinary players can only remember the locations of about half a dozen pieces. But when the pieces are distributed randomly, the experts are no better than the non-experts in placing the pieces. The experts learn to recognize at a glance meaningful spatial patterns - often referred to as 'chunks' by cognitive psychologists - and use these patterns to process information about the game.
- ⁶ Textbooks spend an inordinate amount of time on defining the obvious and making distinctions that are too subtle for elementary school children. For example, one text currently in use in Saskatchewan includes several lessons on defining lines, rays, and line segments and on their symbolic notation. Despite the attempt by the text to distinguish between lines and line segments, a later lesson discusses "lines" of symmetry when "line segments" of symmetry is more properly used. I argue that such pedantry is unnecessary and should be eliminated from the elementary and middle years program.

- ⁷ Hoffer reports that one of the more surprising perceptions of children, including some algebra students, is the claim that a rectangle has just one right angle, namely, the one at the lower left when the rectangle is oriented in the standard textbook position (p. 216).
- ⁸ One activity I used successfully with primary children involved a length of yarn strung through some book rings. Children were arranged in groups of 3 to explore the properties of a triangle. With each member of a triad grasping a ring, they were instructed to carry out the following types of tasks: (1) form any shape other than a triangle (they liked this one); (2) form a 'skinny' triangle; (3) form a 'fat' triangle; (4) form a triangle that looks like this picture; (5) form a triangle that has equal sides; (6) form a triangle with a corner that looks like this picture; (7) form a triangle that has a corner larger than this.
- ⁹ Children should be given the opportunity to read and discuss stories rich in geometric descriptions. Teachers (and children) can also create descriptions such as the following with the aim of identifying the described shapes:

Bill entered the dark cave carefully as he could hear heavy breathing. He held in his hand the rectangular magical charm given to him by the sorcerer. As he walked around the cave, he noticed it was shaped in an odd fashion, having three walls of equal length. He bumped into an object about his size. He found he could place his arms around it from top to bottom. He felt the top. It seemed to be in the shape of a circle. ...

- ¹⁰ I once observed a grade 4 student who became so frustrated with her attempts to form a shape with elastic bands on a geoboard that she eventually refused to work with the geoboard.
- ¹¹ The plane of projection in an *isometric* drawing is equally inclined to the three principal axes of the object, so that all dimensions parallel to these axes are represented in their actual proportions. The sides slope away in two directions at 30 degree angles. *Oblique* drawings slope away in one direction only at an angle of either 30 or 45 degrees.

CHAPTER 13

DATA ORGANIZATION AND ANALYSIS

People confront quantitative data compiled and displayed in varying forms, including lists, tables and graphs. To cope effectively with such information, one must learn various techniques of data management and analysis, especially the collection, organization, displaying, summarization, analysis and interpretation of data. The importance of learning to work with quantitative information has been acknowledged in the NCTM Standards (1989), in the report Everybody Counts (Mathematical Sciences Education Board, 1989) and, most recently, by the National Council of Supervisors of Mathematics (1989). In identifying, what the NCSM termed, "essential mathematics for the 21st century," students need to develop the following competencies:

Students should plan and carry out the collection and organization of data to answer questions in their everyday lives. Students should know how to construct, read, and draw conclusions from simple tables, maps, charts and graphs. They should be able to present information about numerical data, such as measures of central tendency (mean, median, mode) and measures of dispersion (range, deviation). Students should recognize the basic uses and misuses of statistical representation and inference. (p. 29)

Unfortunately, studies of classroom practice indicate that the development of these competencies are generally ignored by classroom teachers and, if they are dealt with at all, they are often considered as "fillers." NACOME (1975), for example, found that about half of the American elementary school teachers sampled in their study either ignored the topics of graphing and statistics or gave them a cursory treatment. Robitaille, O'Shea and Dirks (1982) found that elementary school teachers (grade 8) in British Columbia did not believe it was important to teach simple techniques of statistical analysis. This result contrasted sharply with the importance given to this topic by teachers in other countries. Although textbooks typically include some work on data analysis, it consists primarily of exercises requiring students to read tables or graphs constructed from hypothetical data.

As discussed in Chapter 1, consumers and workers need to acquire the ability to deal with quantitative information. To help children develop this ability, the elementary and middle years mathematics program should provide students with frequent opportunities to collect, organize, tabulate and represent data through simple ex-

perimentation, survey research, and the use of publications that contain realistic rather than contrived quantitative information.

Children should not spend an inordinate amount of time learning to construct graphs, as carefully drawn graphs are rarely done by the average person and, more importantly, the time taken in construction can be better employed in helping children learn to read and interpret graphical information. When graphs are done manually by children, they should typically involve simple sketches. In the later middle grades, children should be taught how to use plotting software to produce more visually appealing graphs and to experiment with alternative ways of representing data.

Activities focussing on data management and analysis involve many other mathematical topics: measurement, reading and writing numbers, counting, computation, and some geometric principles. Thus, children can practice many types of mathematical concepts and skills and possibly begin to see connections between seemingly unrelated topics through the study of this topic.

Teachers need to be provided with activities organized according to the data management and analysis skills involving (1) the collection of data, (2) the organization and display of data, (3) the summarization of data, and (4) the interpretation of the organized and displayed data.

Collecting Data

Data can be gathered from a number of sources including simple experimentation, survey research, published compiled data, and simulations. Teachers will have to be provided with challenging but developmentally appropriate data collection activities for their students.

Simple Experimentation

Students in the elementary and middle grades should investigate the relationship between two measured variables by conducting the following types of simple experiments:

- the number of drips of water exiting from a container over a period of time.
- the growth of a plant over a period of time.
- the length of "stretch" of an elastic band stretched by various weights.
- the heights versus the mass of children in the classroom.
- the changing length of a shadow of a pole throughout the school day.
- the heights of children and the length of their standing vertical jump.
- the changing temperature of hot water cooling over a period of time.
- outdoor temperatures taken over a period of time.
- the circumference versus the diameter of various cylindrical containers.

These types of simple experiments reveal the close connection between the mathematics curriculum and the quantitative aspects of the science curriculum. Students in both subjects should be provided with regular opportunities to practice their mathematical skills by measuring, counting, and calculating and to explore quantitative relationships. Such activities also serve as a good alternative to the commonly seen and seemingly pointless classroom activities such as "Measure the length of your textbook and the width of your desk." Students will begin to realize that real data are "messy" because measurement error cannot be eliminated entirely from these experiments. Later attempts to organize and predict will reveal the messiness of the collected data.

Simple experimentation can also be broadened to include activities typically used to investigate number patterns: for example, investigating the relationship between the number of nickels in a pile and its sum, recording the data in a table and predicting sums based on the information gathered. Simple probability experiments involving die throwing, coin tossing, spinners, and possibly computer simulations can generate interesting data to be organized and later interpreted.

A simple "function machine" is a teaching tool that can be used to stimulate student thinking about number patterns and to make predictions based on these patterns.¹ A function machine usually appears as a cardboard box with two labelled holes: one square-shaped hole labelled "Input," the other triangular-shaped hole labelled "Output." After a mathematical rule such as $\Delta = 7 + \square$ is displayed to the children, the teacher (or a student) writes a number on a square card and enters the card through the input. The children are challenged to determine what number should be written on the triangular card, which will subsequently exit through the triangular output hole.²

Students can record both the input and output numbers in a table and use this information to predict the output number given any input number. Students can be challenged to create different and more complex rules (functions) and predict their output, to predict the input given the output, and to construct a rule given the input and output.

Survey Research

Data gathered through survey research can provide a rich source of personal data that can later be organized and interpreted. Primary children can begin by collecting dichotomous information: e.g., who likes/dislikes a particular food, sport, colour, book, popular song, and so forth. The surveys can be later broadened to include data that can be placed in more than two categories.

Older students should also be encouraged to carry out surveys that model those done by professionals. For example, a survey of the number of cars passing through the

school intersection throughout the school day can help children understand how decisions about traffic planning are often made.

Published Compiled Data

Teachers need to be encouraged to take advantage of published compiled data that can be analyzed and used to stimulate student thinking. The daily newspaper is a rich source of information, providing general graphical and tabular information, weather statistics, business statistics, social statistics, government statistics, polls and surveys, and sport and entertainment statistics.

Magazines available in the home and school are another useful source of quantitative information: Time and Newsweek, Owl, Sports Illustrated, Golf Digest, MacLeans, Discovery, Popular Science, Readers' Digest, and T.V. Guide, to name a few. Some more specialized reference books can provide data that can be used to develop activities for older students: Statistics Canada publications, Guinness Book of World Records, World Almanac, and so forth.

Tom Parker's book In One Day, although listing American statistics, can be used to create quantitative problems. The book provides information on a wide variety of daily statistics including the number of television sets bought in one day, the amount of dirt stirred up on unpaved roads in one day, the amount of freight hauled by trains in one day, and so forth.

Of course, data presented in other subject areas, especially health, social studies, science, and physical education, should be used to develop interesting and useful data organization and analysis activities. The statistics provided in school are ripe for analysis: enrolment, body statistics, test scores, and so forth.

In collecting survey data, children become aware of many important aspects of survey research not apparent when dealing with someone else's data. For example, in designing a survey of student preference, students need to learn how the choice of questions and the selection of the survey participants can influence the outcome of a survey.

Organizing and Displaying Data

The collection of data is obviously only the first step in learning to organize and analyze quantitative information. The next step is to do something with this collected information, usually to organize, display and summarize it.

Data are often organized to communicate information in a concise and visually appealing way. However, as Russell and Friel (1989) maintain, the real user of statistics organizes data in order to "to understand the data during the process of

analysis" (p. 142). They claim that students should be taught about data representation primarily for the following reason: "Representing data in a picture, table or graph is a way to discover the features of the data, to array the data so that their shape and relationships among aspects of the data can be seen" (p. 142).

They advise that students need to learn two important principles about pictorial representation: (1) that a picture of the data is a beginning, not an end; (2) more than one representation of the data can be used. These principles are often overlooked by teachers and textbook authors, and children frequently become more involved in learning the mechanics of graph making instead of learning their various uses.

In the elementary program, children should be introduced to the following types of data displays: (1) bar graphs, (2) tables and lists, (3) timelines and timetables, (4) tally graphs, (5) various coordinate systems. Students in the middle years should work with (6) pie graphs, (7) histograms, (8) line graphs, (9) scattergraphs, (10) stem-and-leaf graphs, and possibly (11) box- and-whiskers graphs. All these displays with the exception of stem-and-leaf and box-and-whisker are frequently seen in some form or other and need no further elaboration.³

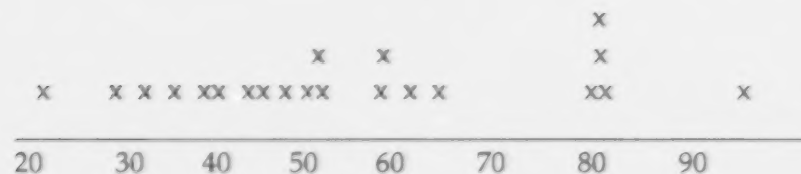
A **stem-and-leaf display** is a type of frequency distribution typically used when data is spread over a larger range. Russell and Friel show how data gathered about the ability of 21 children in holding their breath are arranged in a stem-and-leaf display:

```

2 2
3 0,2,5,8,9
4 2,3,5,7,7,8
5 5,5,9
6 2
7 9,9,9
8 0
9 1

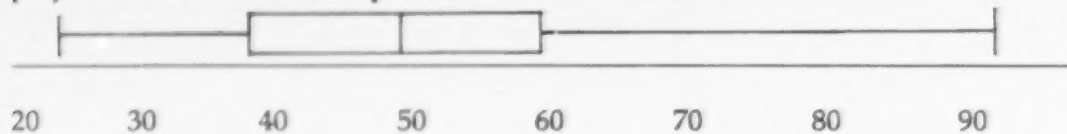
```

The entries in the list are read from top to bottom as 22 seconds, 30 seconds, 32 seconds, and so on. The shape of the data is more apparent in the stem-and-leaf display than in the number line portrayed below:



In a stem-and-leaf display, the median and other quartiles can easily be determined by partitioning the distribution into four equal parts and counting from the least to highest value. Using the data above, the first quartile is calculated to be between 38 and 39 seconds, the second quartile (the median) to be 47 seconds, and the third quartile to be between 59 and 62 seconds.

The more complex **box-and-whiskers** plot can be used to summarize the data displayed in the leaf-and-stem plot:



The box illustrates the range of "breath holding time" for the middle half of the class (the two ends of the box), the median time (the vertical line inside the box), and the range of time for the entire class (vertical lines at the end of the "whiskers"). Thus, an incredible amount of information can be portrayed in a condensed yet quite readable form.

Scatterplots or **scattergrams** can be used to discover relationships between two variables often unnoticed when the data are organized into a table. The scattergram and table presented on the next two pages shows the grams of fat versus grams of protein in individual servings found at various fast food restaurants.⁴

Once students have constructed a scatterplot they can be presented with the following types of questions to stimulate their thinking.

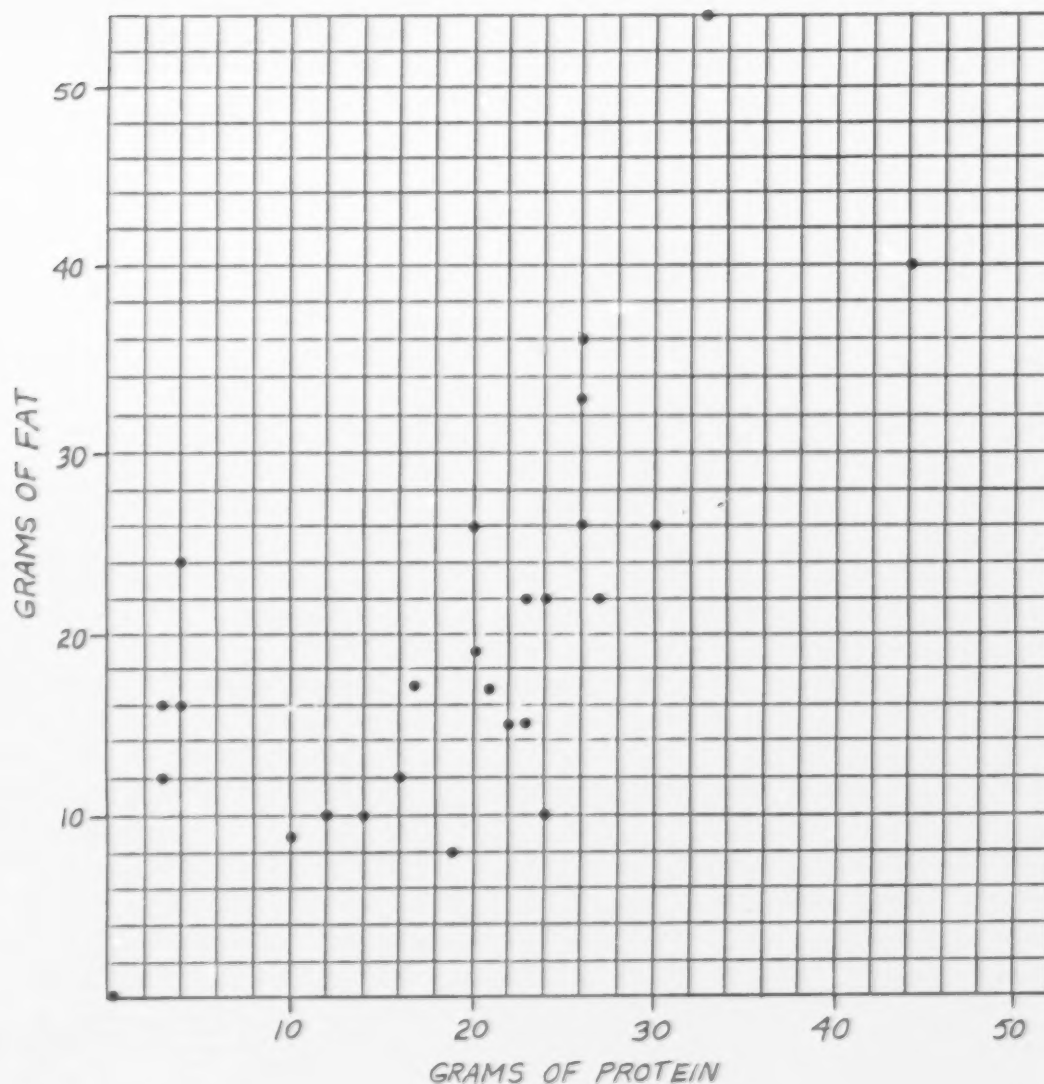
1. Suppose you wish a meal with a high protein but low fat content. By examining the scatterplot, identify your choices. Can you find these items in the table?
2. Locate the items on the scatterplot that you should probably avoid eating? Defend your answer.
3. Is there a positive, negative, or no association between grams of protein and grams of fat?
4. Can you think of any foods, fast food or not, that you could include on the scatterplot to reduce the degree of association?
5. What types of items seem to be outliers and reduce the degree of association?

6. What do the items identified in Question 4 seem to have in common?
7. If a new item has 32 grams of protein, about how many grams of fat would you expect it to have?
8. Can you conclude that the item located at the origin is healthful?
9. Which item has the highest ratio of fat to protein and which item has the lowest ratio? Can the scatterplot be used to identify these items or do all ratios in the table have to be calculated?
- 10.

Make up other questions based on the table and scatterplot.

Protein versus Fat

The following scatter plot shows the grams of fat against the grams of protein in individual servings of lunch and dinner items sold at various fast food restaurants.



The following table lists the items in the previous plot with their grams of protein and grams of fat.

	Protein grams	Fat grams
Big Mac — McDonald's	26	33
Cheeseburger — Hardee's	17	17
Double cheeseburger — Burger Chef	23	22
Cheeseburger w/Bacon Supreme — Jack-in-the-Box	33	54
Single — Wendy's	26	26
Double — Wendy's	44	40
Hamburger — McDonald's	12	10
Quarter Pounder — McDonald's	24	22
Whopper — Burger King	26	36
Roast beef — Arby's	22	15
Beef and cheese — Arby's	27	22
Roast beef — Hardee's	21	17
Big fish — Hardee's	20	26
Ham and cheese — Hardee's	23	15
Thick-crust cheese pizza — Pizza Hut	24	10
Super Supreme thin-crust pizza — Pizza Hut	30	26
Idiot's Delight pizza — Shakey's	14	10
Cheese pizza — Shakey's	16	12
Chicken McNuggets — McDonald's	20	19
Chili — Wendy's	19	8
French fries — McDonald's	3	12
Onion rings — Burger King	3	16
Chocolate shake — McDonald's	10	9
Apple turnover — Jack-in-the-Box	4	24
Chocolate chip cookies — McDonald's	4	16
Carbonated beverages	0	0

Source: P. Hausman, *At-A-Glance Nutrition Counter*, 1984.

Although most visual portrayals will incorporate numerical symbols to represent quantities, objects and pictures can often serve an equivalent purpose. **Object graphs** can be constructed by organizing the actual objects used in the data collection. For example, children can construct a graph by arranging their shoes into various categories: by colour, style and so forth. The children themselves can form part of a graph: arranging lines of children by the month or day of their birthdays, for example. Object graphs are particularly useful for introducing data organization to younger children.⁵

Proceeding from object graphs, children can learn to work with **picture graphs** - using pictures, drawings or even objects such as counters, tongue depressors, multi-link cubes, and paper strips. For example, young children can represent a classroom's breakfast preference by pasting pictures of breakfast items instead of using a number to record the frequencies. Shulte (1982) recounts a graphing activity in which primary children pasted paper outlines of teeth to construct a frequency chart to depict each child's loss of teeth. These simple types of picture graphs lead quite naturally to pictographs, a commonly seen form of data organization.

Multi-link cubes, which can be easily and securely attached, can be used to produce a wide array of visual arrangements, leading, quite naturally, to bar graphs. Materials such as matchboxes, index cards and, most recently, "Post-its" can be used conveniently to represent units of information.

In a revised program, teachers should ensure that children in the upper elementary years are provided with some experience in using computer plotting programs. However, they should not be used until students become familiar with the various forms of data portrayal.

Computer plotting programs allow children to answer many types of "what if" questions not easily answered if graphs must be done manually. For example, what would happen to the general shape of a line graph if each test score is doubled? Or, what would happen to the shape of a pie graph if each amount is doubled? Several suitable plotting programs should be identified and be available for use by teachers and students.

Summarizing Data

After organizing the collected data in some manner, a skilled statistical worker when confronting an abundance of numerical data always attempts to summarize or characterize this information. This is usually done by considering the shape of a distribution, the range of data, and by looking for both typical and atypical (sometimes referred to as outliers) data sets. Russell and Friel believe that perhaps the most important idea in data analysis to communicate to students is to pay attention to the

shape of the organized data. They suggest that teachers emphasize the apparent "clumps, bumps, and holes" and encourage children to ask questions such as the following: "When you look at a table or graph, what strikes you about the data? Where are the data clumped? Are there more than one clump? Are there bumps of data in surprising places? Are there holes that contain no data?" (p. 138).

Eventually children should learn to use and calculate the various descriptive statistics used to describe variation and central tendency. However, in the elementary grades, the approach should be informal and no formulas should be introduced. Again, it should be emphasised that, although children can quickly learn to calculate an arithmetic mean (the calculation requires only a knowledge of addition, counting, and division), a functional and deeper understanding rarely is possible before the later middle years or secondary years. For example, in summarizing the results of a National Assessment of Educational Progress study, Carpenter, Corbitt, Kepner, Linquist and Reys (1981) reported that only 16 percent of thirteen-year-olds could find a missing piece of data when given the average and the remaining data entries, and only 7 percent could perform successfully on items involving the concept of a weighted mean. About 35% of grade 7 students in the 1985 British Columbia Mathematics Assessment (Robitaille & O'Shea) correctly answered the following question similar to the one used by NAEP:

The average age of 4 children is 6 years. If the ages of 3 of the children are 4 years, 8 years, and 3 years, what is the age of the fourth child?
(p. 330)

I suspect that many students are not familiar with the term "mean." Only 21% of grade 10 students in British Columbia could correctly calculate the mean of the following set of numbers: 2, 2, 2, 3, 4, 5, and 10. Clearly, we should ensure that students become familiar with the important statistical notions of mean, median and mode.

Interpreting Data

An important aspect of data management and analysis is ascribing meaning to the collected and organized data. Often teachers limit their attention simply to the reading of the portrayed data (for example, asking questions such as "How many children in our class had cereal for breakfast?") rather than challenging students to identify, interpret and explain underlying relationships. In a revised program, the collected and organized data "should be used to generate new questions, not just to obtain 'answers.' Students should come back to the data for comparison and use them as a resource to solve other problems" (Russell & Friel, p. 135).

Russell and Friel suggest that students need to develop the following habits of thinking that focus on wondering and predicting:

Questioning. Do the data make sense? Are the data appropriate to their intended use? Do the data tell me what I want to know? Is the information reasonable? Is it reliable? Where and how were the data collected? What does this information tell me? What **doesn't** it tell me? Can I represent the data differently and get a different message?

Conjecturing. Perhaps this means that ... If ...then. Suppose ... What if ...? What other questions do the data suggest?

Seeing relationships. What do I already know that the data relate to? What other work has been done - by me, by others, or by the class - that applies here? Does this result have anything to do with the fact that all the people surveyed were in the sixth grade, were boys, or lived in the same town? Does one variable have anything to do with another variable?

Withholding judgement. I wonder why these data don't fit with what I already know? Have I tried enough samples? Does my idea work in all cases? How do I know or what don't I know? How else might I think about the problem and the data I need? What further information is needed?

Building theories. I can generalize the following from what the data show ... I can prove that ... One theory that would explain the shape of the data is ..., but an alternative theory might be ... (pp. 141-42).

The simple experiments listed earlier can be used to introduce these habits of thinking: for example, consider the simple experiment of stretching an elastic band with metal washers. The data can first be organized into a table listing the number of washers and the associated "stretch." When the number pairs are plotted, the shape is generally linear except for the endpoints. Children can be encouraged to explain the shape. They can also be asked to explain a sketch in which the graph becomes very non-linear and ultimately discontinuous. (The point at which the rubber band breaks, in other words). Or alternatively, they can be asked to explain why the graph would not continue endlessly in a straight line. They can also be challenged to use this graph and the elastic band as a simple weight measure. Older children can repeat the experiment using different thicknesses of elastic bands or heavier springs and compare the resulting graphs.

In summary, data organization and analysis needs to be taught in such a way that children are actively involved both physically and mentally. A revised mathematics program needs to provide teachers with sufficient practical ideas to ensure an active

approach is implemented. This important mathematical topic and everyday life-skill cannot continue to be ignored by teachers.

Footnotes

- ¹ The idea of a function and a "function machine" is also discussed in the chapter on algebra.
- ² I used this activity several years ago with primary grade students. The function machine was constructed from an appliance box large enough to room a student. The student had a copy of the mathematical rule and was instructed to write the number on the triangular cards and push the card through the triangular output hole after receiving a square-shaped card inserted through the input.
- ³ John Tukey, an influential statistician, created many new and relatively straightforward techniques of data anaysis, which are discussed in his book Exploratory Data Analysis published in 1977 by Addison-Wesley. These techniques have been incorporated into practical classroom activities in the Dale Seymour Quantitative Literacy Series, which could be used with middle years students.
- ⁴ The graph and table are taken from the Quantitative Literacy Series (pp. 92-93) published by Dale Seymour and reproduced with permission. I have modified the list of questions accompanying the chart and table.
- ⁵ A soon-to-be published series Used Numbers (Dale Seymour) appears to contain many useful data organization and analysis activities: How much sleep does a first grader get? How many raisins are in a box? How many cats make a lion? The six books developed for the elementary and lower middle years are as follows: "Statistics: The shape of the data" (grades 4-6); "Measuring: From paces to feet" (grades 3-4); "Sorting: Groups and graphs" (grades 2-3); "Counting: Ourselves and our families" (grades K-1); "Statistics: Prediction and sampling" (grades 4-6); "Statistics: Middles, means, and in-betweens" (grades 5-6).

CHAPTER 14

ALGEBRA

School algebra is the study of operations and relations among the use of numbers through the use of variables or literal symbols, usually represented by the letters of the alphabet instead of numbers. Algebra also includes the notion of function, the mapping of a number or numbers onto another number.

The power of algebra is derived from its compactness: the ability to represent mathematical rules and laws using symbols and operations. Symbolic formulas are not only more compact than verbal rules, but they also permit manipulations for solving equations and deriving new relationships that would be extremely difficult or impossible without them. Hence, the term "generalized arithmetic" has often been used in the past to describe algebra.

Because of its traditional placement in the later grades, algebra is often considered too difficult or too abstract for students in the elementary and lower middle years. However, it can be argued that students who learn to handle computation should be able to handle many aspects of algebraic manipulations. After all, students likely need as much or more knowledge to carry out the step-by-step procedures required to find the product of two numbers such as 23 and 78 or to add fractions such as $\frac{5}{6}$ and $\frac{3}{4}$ than they need to multiply the two monomials $(x + 3)$ and $(x + 1)$. Given suitable instruction, middle years students can and should learn some fundamental notions of algebra. Some preparatory concrete work with algebra should also be included in the elementary grades. **The emphasis on algebra instruction in the elementary and middle years should be on the development of understanding rather than on technical proficiency, however.**

The Goals of Algebra Instruction

Why should algebra be taught when few students will ever use their learned skills outside of the mathematics classroom? Fey (1989) provides several reasons:

Algebra is a symbol system of unparalleled power for communicating quantitative information and relationships, it is a training ground in careful rule-governed reasoning, its development is a significant thread in the history of mathematics, and its theoretical structure is based on the concepts and principles that generalize to provide organizing schema in nearly every other branch of mathematics. (p. 207)

Fey states that the most convincing justification for the teaching of algebra is its contribution to problem solving in nearly every scientific discipline. He argues that in selecting content "it seems especially important to analyze the ways that algebraic ideas and methods aid in reasoning in various problem situations" (p. 207).

Usiskin (1988) argues that algebra is more than "generalized arithmetic" and more than a vehicle for solving certain types of problems. He views algebra as the means to describe and analyze relationships and the key to the characterization and understanding of mathematical structures. He states, "Given these assets and the increased mathematization of society, it is no surprise that algebra is today the key area of study in secondary school mathematics and that this preeminence is likely to be with us for a long time" (p. 18).

Leitzel (1989) argues that for most students the study of algebra should have priority over the study of other traditional topics: "We must set aside the notion that the gateway to mathematics is through the rational numbers. We are sacrificing thousands and thousands of students every year on the altar of fractions when they could be involved with meaningful mathematics" (p. 31). Leitzel implies that in an overcrowded curriculum algebra should replace some traditional topics, especially those topics that are relatively unimportant in a mathematical or practical sense or that have been obsolesced by calculating technology.

What should be the specific goals of algebra instruction? Thorpe (1989) identifies several goals:

1. To develop skills in the solution of equations, finding numbers that meet specified conditions.
2. To teach students to use symbols to help solve real problems, such as mixture problems, rate problems and so forth.
3. To prepare students to follow derivations in other subjects, for example, in physics and engineering.
4. To enable students to become sufficiently at ease with algebraic formulas that they can read popular scientific literature intelligently.

He warns that "algebra should not be taught as a collection of tricks, as is common in textbooks - a trick for this, a trick for that. Students should see algebra as an aid for thinking rather than a bag of tricks" (p. 12). He proposes that "students should grow in their natural powers of seeing the mathematical elements in a situation, reasoning with these elements to come to relevant conclusions, and carrying out the process with confidence and responsibility" (p. 12).

The Understanding of Algebra

What specific aspects of algebra should be taught in the elementary and middle grades? The formal study of algebra in Saskatchewan has begun traditionally in grade nine. Except for the development of a few formulas and the solution of simple open sentences, little algebra content is included in the earlier middle grades. In a revised curriculum, students should be exposed to some aspects of algebra beginning in grade 6. Rather than focussing on technical skills involving symbol manipulation, however, the program should emphasise the understanding of the fundamentals of algebra. By the end of grade nine, students should have been exposed to the following aspects of algebra:

1. Variables in equations, formulas and in functions.
2. Intuitive methods for solving word problems.
3. Informal methods for solving equations and formulas.
4. The representational forms of functions.
5. Equivalent algebraic expressions, some simple rules of algebra.

Variables

To the mathematically sophisticated person, a variable appears to be a straightforward concept. But several writers argue that a variable is a more difficult concept to understand than has been believed by many mathematics educators and teachers. Schoenfeld and Arcavi (1988) list 10 definitions of the term variable taken from the literature varying from the simple form "Latin - variabilis: 'changeable'" to the more complex form listed below:

A variable is a symbol that can be replaced by any element of a designated set of numbers (or other quantities) called the domain of the variable. Any member of the set is a value of the variable. If the set has only one member, the variable becomes a constant. If a mathematical sentence contains two variables related in such a way that when replacement is made for the first variable the value of the second variable is determined, the first variable is called the independent variable, and the second is called the dependent variable (p. 421).

They suggest that the many definitions can be condensed to two forms, a duality originally proposed by Freundenthal. The term "polyvalent" refers to statements such as " $a + 3 = 3 + a$," which holds true for all quantities substituted for a . The term

"variable object" refers to statements such as " x approaches 0", where x represents a continually increasing or decreasing value.

Usiskin argues that most students think all variables are letters that stand for numbers. He states that this is a narrow view because a variable can represent points in geometry, propositions in logic, matrices and vectors in linear algebra. He claims that because "variables have many possible definitions, referents and symbols, trying to fit the idea of a variable into a single conception oversimplifies the idea and in turn distorts the purposes of algebra" (p. 10).

A large scale study of 13- to 16-year-old students in the United Kingdom identified a number of misunderstandings about algebra and the meaning of a variable (Booth, 1988). Students were often confused that algebra, unlike arithmetic, does not always produce a particular numerical answer. For example, given the problem to determine the distance travelled by a spaceship in y stages of 11 light years each, one student was guided into writing the statement $11 \times y$ but, after the interviewer said "Fine," she responded, "What, is that all it was? Why didn't you say so? I thought you wanted the answer" [referring to a number] (p. 21). Another student who correctly represented the perimeter of a shape constructed from adjacent lengths of 5 units stated, "Well, you can't give a proper answer, because you don't know what 'n' is. If I knew 'n', I could work it out, but as it is, all you can put is $5n$."

Some students respond literally to variables represented by letters of the alphabet. Booth provides an example of a 15-year-old confused by the instruction "Add 3 to $5y$." He stated that 'y' is a letter standing for something: in this case, "eight lots of y." When asked to provide a possible referent for y , he produced only words beginning with the letter y : for example, yachts, yams, and yoghurt. He also stated that the referent "... would have to start with the letter y , 'cause you've got a letter y there. So you need a y for the start of the word" (p. 27). Booth points out that the students' literal view of a variable is perhaps not too surprising considering that their initial exposure to variables involves formulas such as $A = l \times w$, where A , l and w are the beginning letters of the referents area, length and width, respectively.

Booth also found that students tend to view variables as representing a single numerical value rather than a multitude of possible values. She argues that this limited view likely results from the early experiences with equations such as $x + 3 = 8$, in which no doubt exists about the value represented by the variable x . She found in her study that students who hold this narrow view often assume that different letters must therefore stand for different numerical values in equations such as $x + y + z = x + p + z$. One fifteen-year-old interviewed in the study surmised that the variable y in the aforementioned equation could not equal p because each letter represented a different amount. When asked to explain, he commented, "Well, I've always found they're different. I've never come across one where they're the same amount" (p. 28).

Booth comments that, although students had been exposed to linear graphs such as such as $y = x$ where different letters represent equal numbers, they frequently respond that, in this case, y could equal x because "this is graphs, not algebra" (p. 29).

Kieran (1989) claims that students' understanding of a variable progresses through the following stages:

1. Letter evaluated: The letter is assigned a numerical value from the outset.
2. Letter not used: The letter is ignored or its existence is acknowledged without giving it a meaning.
3. Letter used as an object: The letter is regarded as a shorthand for an object or as an object in its own right.
4. Letter used as a specific unknown: The letter is regarded as a specific but unknown number and can be operated on directly.
5. Letter used as a generalized number: The letter is seen as representing, or at least being able to take on, several values rather than one.
6. Letter used as a variable: The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values (p. 42).

If students are to understand the concept of a variable, they must be exposed regularly to equations, formulas, generalizations and functions, especially those that students have developed by themselves. The program should consider the developmental levels of understanding of a variable and plan instructional activities accordingly.

In the elementary grades students should occasionally use letters or place holders to represent variables and learn to substitute numbers for a given variable. These understandings should be developed through the use of concrete materials, of course. For example, an expression such as $3b + 2$ can be represented by 3 small boxes plus 2 counters. Children can place an equal number of counters in each box beginning with 1, count the total and place the results in an organized table or chart:

Number of counters in each box b	1	2	3	4	5	6	7
Total number of counters	5	8	11	14			

The students would be challenged to find and express a pattern in their own words. Older elementary students can be shown how to plot such information on a graph.¹ The NCTM Standards makes the following recommendation about algebra in the elementary grades, which they include under the heading of "Pattern and Relationships":

Physical materials and pictorial displays should be used to help children recognize and create patterns and relationships. Observing varied representations of the same patterns helps children identify its properties. The use of letters and other symbols in generalizing descriptions of these properties prepares children to use variables in the future. This experience builds readiness for a generalized view of mathematics and the later study of algebra. (p. 60)

Both elementary and middle years students should be provided with frequent opportunities to work with activities designed to help them connect diagrams, graphs, and verbal statements with algebraic expressions involving a variety of different variables.²

Working with simple computer languages such as BASIC and Logo might be helpful in developing students' understanding of a variable. Some empirical evidence is available to support the claim that programming experiences can provide students with a conceptual basis for variables (for example, see the study of Logo reported by Sutherland (1987)). Whether the time taken to learn programming can be justified by the amount of mathematics learned incidentally through programming remains an unanswered and debateable question. It could be argued that the way variables are handled in computing languages by assigning values to a memory location (e.g., $A = A + 1$ is interpreted as add one to the current value of A and store this new value in the memory location assigned to A) might confuse rather than aid students.

Spreadsheets can also be helpful in developing the notion of a variable.

For example, the area of a rectangle can be calculated by entering the formula $+ B2 * B3$ into the cell B4, where the cells B2 and B3 contain the values for the length and width. When the value in B2 or B3 is changed, the change in area is calculated automatically. As another example, if cell A1 contains the value 1 and if cell A2 contains the formula $+ A1 + 1$, a value of 2 will be located in cell A2. It should be noted that variables in spreadsheet software represent cell locations containing the current values rather than the values themselves. In this way, spreadsheets can be made to count by ones when the formula is copied down column A.

Word Problems

As in arithmetic, word problems in algebra are designed primarily to help students learn to translate verbal statements into algebraic expressions. Few mathematics educators believe that solving algebraic word problems contributes to an acquisition of generalized problem solving processes. Thorpe (1989) argues that most word problems appearing in textbooks are contrived rather than real world applications of algebra. He argues that, although the typical problems classified as coin, mixture, age, consecutive integer, and so forth, do lead to algebraic equations to solve, their frequency misleads students about the value of algebra. Usiskin (1980) expresses a similar viewpoint that current word problems are so contrived that they succeed only in teaching students that algebra "has no applications, that you should not use common sense arithmetic to solve [problems], and that algebra is difficult" (p. 416).

I believe that under certain conditions that even contrived word problems have their place: (1) if they are considered by students and teachers to be "puzzles" rather than "problems;" (2) if they are solved in a variety of ways; and (3) if the artificiality of the problems is discussed with the students with the goal of producing more realistic alternatives. For example, when students are presented with a typical mixture problem they should learn that tables or some other aid would more likely be used than algebraic methods to solve similar problems in the real world. However, the students should see how algebraic reasoning is often used to develop the tables in the first place.

Elementary and middle years students are capable of solving some rather difficult "algebra problems" if they are encouraged to use informal methods of solution. Sawyer (1989) discusses how diagramming can be used to help students arrive at an understanding of algebra. He presents the following puzzle:

A man has twin sons. The twins are the same height. If we add the height of the man to the height of one son we get 300 cm. The total height of the man and two sons is 420 cm. What are their heights? (p. 6)

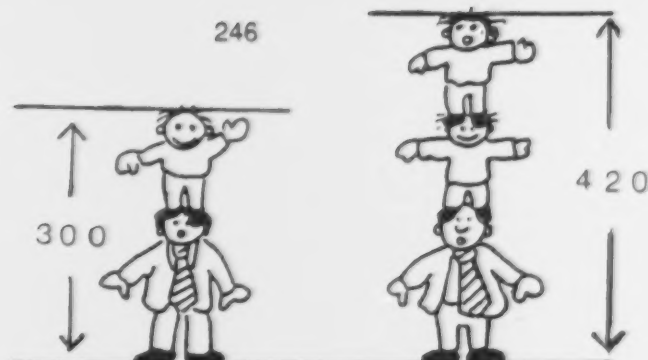
Sawyer suggests that each statement be connected with diagrams representing the heights of the father and sons as illustrated below.³ The solution of 120 cm for the height of each twin becomes apparent by comparing the heights and subtracting 300 cm from 420 cm. The height of the father is easily obtained by subtracting 120 cm from 300 cm.

$$m + s = 300$$

$$m + s + s = 420$$

$$s = 120$$

$$m = 180$$



Sawyer claims that he has presented this problem to disadvantaged 8- and 9-year-olds. They not only solved this problem but many other problems involving the "solutions of simultaneous linear equations." Sawyer comments that in presenting such problems "we do not confront learners with abstract symbols x and y , which they are required to manipulate correctly. We start with a problem which they are able to solve by everyday common sense. Only after this do the symbols, m and s , come in as convenient abbreviations for something already understood" (p. 7). (See also Simon & Stimson (1988) for further discussion of solving algebra problems through the use of diagrams.)

Equations and Formulas

Children in the elementary grades are exposed to equations of the form $a * b = c$, where $*$ refers to an operation, and a and b are typically given and c is unknown. (Usually the unknown is represented by a placeholder such as a square or question mark rather than a letter.) Equations are usually referred to as "open sentences" in the elementary grades. As children become familiar with the four operations and their interrelationships, they are expected to solve "missing addend" or "missing factor" equations in which either a or b is unknown.

When given an equation such as $4 + ? = 20$, some children do not use algebraic methods such as transposition to determine a solution. Instead they tend to use substitution or guess and check as the primary method of solution. Other students tend to follow the procedure included in most texts: that is, to rewrite the equation using an inverse operation (e.g., $20 - 4 = n$). Consequently, early formal work with open sentences does not necessarily contribute to an understanding of equation solving nor do students approach equation solving in the manner intended by teachers and textbook writers.

A study by Whitman reported in Kieran (1989) examined the relationship between formal and informal techniques of equation solving with grade 7 students. Students were taught one of three techniques:

1. Intuitive techniques: For example, students in solving the equation $69 - 96 / (7 - a) = 37$ were taught to think "69 minus what number gives 37?" [32]; "96 divided by what number equals 32?" [3]; "7 minus what number equals 3?" [4]. The solution equals 4.

2. Formal techniques: Multiply both sides by 7 - a, remove parentheses and combine terms on each side, subtract 259 from both sides, and so forth.

3. Intuitive techniques followed by formal techniques.

Students who learned only intuitive techniques performed better than those who learned both techniques in close proximity, while students who learned only the formal technique performed worse than those who learned both techniques. Whitman reported that formal techniques appear to thwart students' intuitive ability to solve equations (Kieran, p. 49).

Kieran (1988) identified two different approaches to solve equations used by grade seven students prior to formal instruction: one she termed the "algebraic" approach, the other, the "arithmetic" approach. Students not only preferred one approach over another but viewed the meaning and significance of the various parts of an equation differently (p. 92).

Kieran questioned the students on their understanding of the parts of an equation, its solution, and the equivalence of equations. When presented with an equation such as $2c + 15 = 29$, for example, the arithmetic group viewed the letter "c" as "standing for some unknown number." In other words, the letter was viewed a part of the numerical relationship of the equation. A typical response from students holding an arithmetic view might be that the equation can be thought of as "Doubling the number c, plus 15 more equals 29."

The letter for the algebra group appeared to have meaning only when its value was found: In the case of the aforementioned equation, it had to be transformed into the alternative expression $c = (29 - 15)/2$ before it acquired meaning. Kieran found similar differences between the two groups in the way they viewed the solving of an equation and the equivalence of equations.

She also found that the two groups responded differently to a teaching experiment focussing on solving equations by performing the same operation on both sides of the equation. The algebra group did not appear to understand this "balancing" procedure and preferred instead to use transposing whenever possible. Kieran reports that by the end of the teaching procedure only students in the arithmetic group were regularly using the balancing procedure to solve equations.

In discussing the implications of her study, Kieran criticizes the current approach to solving open sentences seen in elementary school: for example, solving $n + 3 = 7$ by referring to the corresponding inverse equality $7 - 3 = n$. She argues that this approach leads to the view of a variable being the result of a set of inverse operations - with the

letter having no existence of its own within the given equation. She makes the following suggestion:

Elementary experience with placeholders ought to include a conceptual approach to equations like the one adopted in the study, one where the emphasis is not on giving meaning to the letter by transposing the given equation but rather on giving meaning to the letter as a number within the given sequence of operations of the equation. Consequently, finding the value of the placeholder in an open sentence is accomplished by the substitution method. This kind of experience at the elementary school level might help enhance the beginning algebra student's ability to make sense not only of the solving procedures taught in high school but also of equations themselves. (p. 96)

Kieran also criticizes another popular instructional approach. She suspects that the "Think of a number" activity (e.g., I am thinking of a number. If I multiply it by 4 and add 2, the result is 50. What is my number?) can lead to difficulties in algebra. By teaching students to undo the given operations in reverse order to find the original number, it may simply reinforce the limited notion of a letter such as that held by the algebra group.

Kieran argues that the construction of meaning for the balancing procedure is a learning process that takes time. She urges teachers to take into account the likelihood that some of their beginning algebra students will prefer the techniques used by the algebra group, while others will prefer those of the arithmetic group. Consequently, instruction that provides for both preferences will probably be more successful than instruction emphasising only one technique (p. 96).

In the middle years, students should learn a number of informal techniques to solve equations including substitution, cover-up, undoing, balancing, and possibly transposition. The intent is to familiarize students with a broad set of techniques having an intuitive appeal. Mastery of a single equation solving technique should be left to the secondary grades.

Substitution (sometimes referred to as the guess-and-check or generate-and-evaluate method) should be used regularly in the earlier middle grades. Whenever possible, the equations to be solved should be associated with a problem: for example, the "chicken and pigs" problem discussed in the chapter on problem solving or the "father and twins" problem discussed earlier in this chapter.

The use of a calculator permits students to solve not only straightforward linear equations but much more complex equations such as following:

$$(3x^2 + 1) = 2.3$$

Students should be encouraged at first to use numbers at random but adjust their trial numbers through continually refined estimates. For example, an initial guess of 0.5 for x in the above equation led to an answer of 1.75. Consequently, a higher estimate for x was needed. Subsequent trials with 0.6, 0.7, 0.65 and 0.64 led to a result of 2.2288, which is close enough to 2.3, and, consequently, x was determined to equal approximately 0.64. As students gain confidence with this technique, they should learn to become more efficient in selecting a first trial number: "The 1 on the left-hand side means that $3x^2$ is about 1. So x^2 must be around $1/3$. I'll try the square root of $1/3$ as my first number, which according to my calculator is around 0.58."

This activity can be incorporated into a game-like setting such as the following:

Pairs of students are given a calculator and several equation cards. One student selects a card and then the other student tries to solve the presented equation by entering a trial number. The students alternate entering numbers until they determine the value of the variable to within a pre-determined range. Points can be awarded based on the complexity of the equations.

Students should also be exposed to and discuss equations that have no solution (no real roots) such as the following: $(3 + 2/x^2) = 1$. We should encourage the following type of reasoning: "3 is larger than 1, so $2/x^2$ must be negative. But since x^2 can never be negative, the equation can't be solved." Later they can be asked to use this form of reasoning to identify equations with no solutions without resorting to a calculator.

Although the development of informal methods of equation solving should be a major goal of algebra instruction in the middle years, teachers should be aware that informal methods cannot be easily applied to all forms of equations. For example, Ekenstam and Nilsson (1979) found that, although over 80 percent of the top 20 percent of 16-year-olds in a Swedish study could solve the equation $30/x = 6$ correctly, only about half solved the structurally similar equation $4/x = 3$. The first equation can be solved easily through substitution, while the second equation cannot.

Studies have shown that students experience difficulty in considering multi-term expressions as a single unit and do not recognize the structural similarities of equations such as $4(2r + 1) + 7 = 35$ and $4x + 7 = 35$ (Kieran, 1989, p. 45). The **cover-up** technique is intended "to break the stereotype that an unknown is a single letter and to arrive at a broader generalization that an expression containing a variable might

also be treated as an unknown" (Bernard & Cohen, 1988, p. 101). Cover-up refers to an approach in which students are taught to consider any expression containing a variable as a single unit by "covering-up" the entire expression and reconsidering the relationship between the expression and the entire equation. For example, in the equation, $4(2r + 1) + 7 = 35$, the terms would be covered up in a series of stages each invoking a different reasoning procedure:

- Covering the expression $4(2r + 1)$ leads to "What number plus 7 equals 35?"
- Covering the expression $2r + 1$ leads to "What number multiplied by 4 equals 28?"
- Covering the expression $2r$ leads to "What number plus 1 equals 7?"
- Covering the expression r (the variable) leads to "What number doubled equals 6?"

Although the **undoing** and cover-up methods are closely related, each is distinctive enough mathematically and conceptually to be treated as a separate technique. As the name of the method implies, equations are solved by undoing an operational step by using an inverse operation. Often the technique is developed through the use of "flow charts" or analogies such as "gift wrapping."

After using a flow chart to express an equation, students can be taught to undo each step by proceeding through the chart in "reverse" order as illustrated in the diagram below: Solve $10(2x^2 + 1) = 2010$

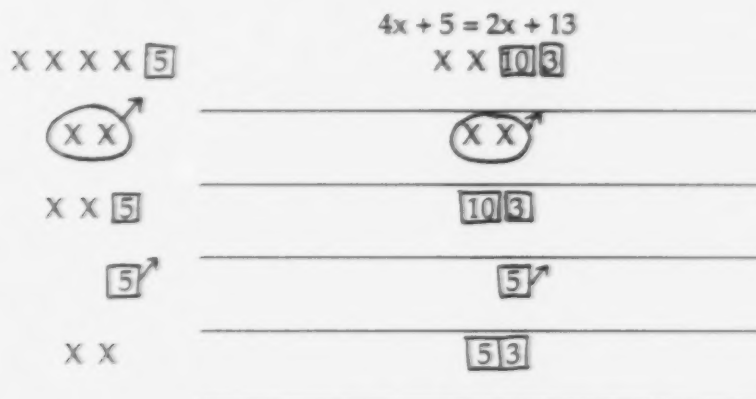
Flow chart: $x \longrightarrow \text{square} \longrightarrow x^2 \longrightarrow + 1 \longrightarrow \times 10 \longrightarrow 2010$

Reverse chart: $2010 \longrightarrow \div 10 \longrightarrow - 1 \longrightarrow \div 2 \longrightarrow \text{square root} \longrightarrow x$

Students can be taught to undo equations by considering each term in the equation as a layer of gift wrapping. First, the "gift" (the numerical value of the variable) is thought of as being "wrapped layer by layer" by carrying out successive operations. The gift is then unwrapped layer by layer by carrying out a series of inverse operations.

The preceding techniques can be used to solve almost all equations presented to middle years students. These techniques are best applied to equations in which the variable appears on only one side of the equation. Other techniques such as balancing and transposing are needed to solve equations such as $3x - 5 = 2(x - 1)$.

Henry Borenson (1985) has developed a series of activities Hands-On Equations that, if used judiciously, appear to have some use in the middle grades to develop the **balancing** method of solving the equation.⁴ The system uses pawns and number cubes and a laminated picture of a two arm balance. Students place the cubes and pawns on the balance to represent an equation. They are taught to make "legal moves" to "balance" the equation. The diagrams below outline the solution to the equation $4x + 5 = 2x + 13$ (X represents a pawn; a number placed in a square represents a cube):



So $x = 4$. The check in the initial setup reveals that $21 = 21$.

Algebraic Expressions

Most adults who have any recollection of algebra remember the seemingly endless instruction on ways to manipulate algebraic expressions; one learns to add, subtract, multiply, and divide them; to factor and simplify them; to collect like terms; and so forth. Often students are given no reason why these manipulations should be done in the first place: for example, why would anyone need to simplify an expression such as $3(x + 4) - 6x + 8$? Clearly, the need to simplify as with other types of manipulative techniques depends entirely on what one plans to do with the results. Unfortunately, students are given few opportunities to apply algebraic manipulations to solve any problems.

Davis (1989) compares the teaching of algebra, especially the current emphasis on learning manipulative rituals, to the way he was taught handwriting:

We practiced circular movements and made O's, we practiced looping motions and made I's and so on. I did learn to write, but this process never struck me as sensible. I never really knew what I was supposed to be doing. (p. 117)

Thorpe (1989) argues that expressions are not important in themselves. They are only important when they are implicitly or explicitly part of an equation or function. He

and many other writers believe that curriculum developers need to re-examine the rationale for insisting on a high level of competence in symbol manipulation considering the inevitable availability of technology designed to carry out algebraic manipulations. He believes that "just as the goal of arithmetic instruction needs to shift away from skill with complex arithmetic operations and toward skill with estimation, mental arithmetic, and approximation, so also must the goal of algebra instruction shift away from skill with algebraic manipulation and toward conceptual understanding and problem-solving skills (p. 19).

What manipulative skills should be learned and which should be presented in the middle grades? I recommend that in the middle years we expect students to be able to perform manipulations such as the following:

- simplifying polynomial expressions by adding and subtracting.
- simplifying polynomials by multiplying and dividing a monomial by a monomial, multiplying a monomial by a polynomial, and dividing a polynomial by a monomial.

I suggest that a formal study of the more complex manipulations such as multiplying and dividing several polynomials and the various techniques of factoring be postponed until the secondary grades.

A more important skill to be emphasised in the middle years is the translation of verbal statements to algebraic expressions and conversely. This appears to be a difficult notion for most children, judging from the results of the 1985 British Columbia Mathematics Assessment (Robitaille & O'Shea). Grade 7 students were asked the question "Which one of the following expressions represents twice a number less 5?" About half the students correctly selected the expression $2x - 5$. Grade 10 students were given a similar question: "Which expression represents a number that is 9 more than two-thirds of a given number." Less than half selected the correct choice

$\frac{2}{3}z + 9$. The most common choice was the expression $\frac{2}{3}(z + 9)$.

Functions

Most people are unaware of being exposed almost daily to functions as they work with interest rates, sales tax, conversion tables and so forth. Because of its many applications in and outside the field of mathematics, many mathematics educators believe that the concept of a function is one of the most important understandings to be acquired by all students. Thorpe (1989) argues that "Functions are basic - in algebra, in mathematics. Let us not teach about functions in algebra; let us make functions the centrepiece of algebra instruction" (p. 18).

In the middle years, students should be exposed to the most intuitive notions of a function. They should learn that a function (whether they learn the term or not is unimportant) is a rule or mapping that assigns to each member of one set a member of another set. Whenever possible, the practical applications of functions should be studied. For example, studying the relationship between the price of an item and the corresponding sales tax: in other words, learning that sales tax is a "function" of price. They should also learn to represent functions as ordered pairs, as entries in a table, and as graphs. As Thorpe (1989) urges, "we should teach the most intuitive and practical definition and not confuse our students with unnecessary abstractions" (p. 13). Certainly, the abundance of terminology associated with functions such as domain, range, and image set, and so on should not be introduced in the middle years. The standard symbolism for a function $f(x)$ does not need to be introduced either.

Research indicates that students find it difficult to connect the various ways that functions can be represented. For example, students may be able to deal with functions as represented by a "function machine" with numerical inputs and outputs from which they have to discover the arithmetic operation performed by the machine or as represented by a table of ordered pairs. But the results of several studies including the NAEP assessment (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981) demonstrate that competency with an implicit form of a function does not necessarily transfer to another more explicit form.

In the NAEP study, students who had studied algebra for two years had difficulty interpreting the standard function notation. Although 98 percent could evaluate $a + 7$ when $a = 5$, only 65 percent could evaluate $f(5)$ given $f(a) = a + 7$. They also had difficulty connecting a table with a simple function. A table of x and y values had several missing y values and students were asked to determine the missing values. Although about 81 percent recognized the pattern of adding 7 to each value of x , approximately one quarter of the students could not generate the function $y = x + 7$. As Herscovics (1989) comments, "Even when the rule for relating two variables has been discovered, many students cannot formalize it into an algebraic expression" (p. 76).

Similar results were found in the 1985 B.C. Mathematics Assessment. About one third of grade 10 students could not correctly match the table of data below with the expression $b = 2a + 4$:

a	b
1	6
2	8
3	10
4	12

NAEP also investigated whether relating the two variables to more relevant situations than a function table would reveal a better understanding. Students were provided with the equation $W = 17 + 5A$, relating the weight of young boys to their age. The following question was posed: "According to this formula, for each year older a boy gets, how much more should he weigh?" Students were presented with the following choices: 5, 17, 22, or 'I don't know.' As Herscovics (1989) writes:

Quite suprisingly, formal instruction in algebra seems to have had little effect on success rates (54% for 13-year-olds vs. 58% and 64% for 17-year-olds with one or two years of algebra, respectively.) Of those students with two years of algebra experience, 30% did not even bother to substitute values for A; they simply read off 17 in the formula or added the two numbers (17 and 5). Since these students apparently did not perceive the relationship between the two variables, one has to question seriously whether they reviewed the formula as a function, even in the most general sense. (p. 76)

Some evidence exists that even college level students experience difficulty in translating tables to functions. For example, only 42% of the engineering students participating in one study could solve the following problem (Lochhead & Mestre, 1988, p. 129):

Weights are hung on the end of a spring and the stretch of a spring is measured. The data are shown in the table below:

Stretch S (cm)	Weight W (g)
3	100
6	200
9	300
12	400

Write an equation that will allow you to predict the stretch (S) given the weight (W).

Lochhead and Mestre found that students had the following difficulties: (1) translating a sentence stating a relationship between two variables into a function (students typically wrote the reverse of what they intended); (2) translating tables expressing a relationship between two variables into a function; (3) translating a linear function into meaningful sentences; and (4) translating a pictorial situation representing a relationship between two variables into a function. Lochhead and Mestre found two types of errors to be noteworthy. First, students tended to perform a left-to-right word-order match when matching statements with variables: e.g., expressing a statement such as "six times as many students as professors" as ' $6S$ ' and writing an equation $6S = P$ instead of $P = 6S$. Second, as has been mentioned earlier in this chapter, students confuse the distinction between variables and labels. Thus, the symbols S and P are interpreted as 'students' and 'professors' rather than as variables representing the "number of students" and the "number of professors" (p. 130).

Lochhead and Mestre comment on the implications of their study:

What is terrifying about these data is that the students sampled were mathematics, science, and engineering majors enrolled in some fairly prestigious universities across the country. We would like to think that people who will be building our future bridges, computers and airplanes would at least be able to perform flawlessly in the simple algebraic problems [presented in a chart] What is most lamentable to us is that our educational system does not seem to address the conceptual issues that would help students overcome these types of misconceptions. (p. 130)

To help prepare students in the middle years for more formal work with functions in the secondary years, they should be provided with instruction emphasising the connection of the various representational forms of functions. Simple experiments involving the relationship between two variables such as those listed in the chapter on data organization and interpretation can be used to collect data in the form of table and then to develop corresponding relational statements (e.g., the elastic band stretches 3 cm for every 2 washers), and finally to write a function (e.g., $2S = 3W$, or $S = 3/2W$, where S = stretch in cm, W = weight in washers).

Studies have shown that it is easier for students to manage functions given in graphical form than in algebraic form. Consequently, Markovits, Eylon, and Bruckheimer (1988) suggest that much more work in the early development of function concepts should be done in the graphical form (p. 58). Because students often have the misconception

that every function is linear (see Markovits et al.), experience with non-linear functions should also be provided.

Students should also be provided with experience with spreadsheets to generate tables developed from a function and to use computing software to graph functions. Whenever computing technology is used, students should be involved in activities that promote generalizing and predicting such as the following:

Plot the following: $Y = 2X$
 $Y = 2X + 1$
 $Y = 2X + 2$
 $Y = 2X + 3$

Predict the graph of $Y = 2X + 5$.

Predict where the graph will intersect the Y axis and the X axis.

Predict which graphs will be parallel to the first four graphs:

$Y = X + 2$, $Y = -2X + 6$, $Y = 2X \times X + 6$, $Y = 2X - 50$

Plot $Y = 2X + 6$ and $Y = (4X + 12)/2$. Explain why the graphs look the same.

Create a function (algebra rule) that goes through the origin. Plot to check your answer.

Predict which of the following graphs is the "steepest":

$Y = 3X + 50$, $Y = 10X$, $Y = 100X - 1000$, $Y = X$

To summarize this chapter: Algebra instruction should begin in the elementary grades and continue throughout the middle and secondary grades. The emphasis throughout the elementary and middle years should be on the development of conceptual understanding rather than the acquisition of standardized, algebraic recipes. As Usiskin states, algebra should be considered a vital component of a revised mathematics program:

Algebra remains a vehicle for solving certain problems but it is more than that as well. It provides the means by which to describe and analyze relationships. And it is the key to the characterization and understanding of mathematical structures. Given these assets and the increased mathematization of society, it is no surprise that algebra is

today the key area of study in secondary school mathematics and that this preeminence is likely to be with us for a long time. (1988, p. 18).

Footnotes

- ¹ I once used this activity with grade 4 students. I constructed a coordinate system on a large peg board. Coloured golf tees were used to represent the ordered pairs we developed through the "box" activity. Most students had little problem in learning how to place a tee in the correct pegboard location. The students were intrigued by the graphs of "expressions" such as $2b$, $2b + 1$, $2b + 2$, and so forth and were able to predict the graph of $2b + 3$ without resorting to their counters and boxes.
- ² See Chalouh & Herscovics (1988) and Simon & Stimson (1988) for sample activities appropriate for middle years students and Thompson (1988) for ideas for teaching algebra to younger children.
- ³ Special thanks to Jessica Latshaw for her willingness to produce this diagram.
- ⁴ I have reservations about any materials that make such grand claims as "It is thrilling to see the joyful expressions on the faces of third and fourth graders as they easily solve such equations as $2(x + 4) + x = x + 16$. They know that this kind of mathematics looks important. Hence, when they experience success with it, their sense of pride and self-esteem with respect to mathematics is greatly enhanced." Many students in the 1960s solved such difficult equations using Cuisenaire rods. The question is: Why do we want grade 4 students to solve these equations when there are more important mathematical ideas to be learned at this level. However, the materials do have some value if they are used cautiously.

CHAPTER 15

THE CURRENT SECONDARY MATHEMATICS PROGRAM

The current secondary program is an amalgam of several different courses. The present courses and their respective enrollments in 1987-1988 are presented in Table 1. Before I propose any program changes, I will briefly review the underlying aims, content, and sequence of these courses in the current Saskatchewan secondary mathematics program.

As in many provinces, the Saskatchewan secondary program tries to accommodate for the vast individual differences in students' abilities and interests in mathematics through a differentiation in the course offerings. The sequence of courses, Algebra 10, 20, and 30, and Geometry-Trigonometry 10, 20, and 30, is often referred to as the "Academic Program." These courses are intended to prepare students for further

Table 1. Student Enrolment in the Secondary Mathematics Programs 1987-1988.

Courses	<u>Enrolment</u>		
	Grade 10	Grade 11	Grade 12
Modern Algebra 10, 20, 30 ^a	13157	11430	10268
Mathematics 10, 20, 30	888	840	645
Geo-Trig & Geometry 10, 20, 30 ^b	7882	5254	7101
General Mathematics 10, 20, 30 ^c	1842	1428	594
Calculus 30	-	-	1289
Total enrolment in each grade	15065	13947	14278

Source: Saskatchewan Education, Evaluation and Student Services

^a Includes Modern Algebra 10A, 20A, 30A (Enriched), and Algebra (French)

^b Includes Geometry 10, 20, and 10A, 20A and Geo-Trig (French)

^c Includes General Math 11, 21, 31 (Modified)

study in mathematics in post-secondary institutions. The other sequence of courses, Mathematics 10, 20, and 30, is often referred to as the "Alternative Program" and is intended to meet the needs of students who do not want to take additional post-secondary courses in mathematics, although they may intend to enter a post-secondary program of study. Only these two programs will be reviewed in this chapter. I have not reviewed the "General Mathematics" courses or Calculus 30 because they are local electives rather than provincial offerings.

The Academic Mathematics Courses

The academic mathematics courses are Modern Algebra 10, 20, and 30 and Geometry-Trigonometry 10, 20, and 30. As can be seen from examining Table 1, the Modern Algebra courses and their variations are the most popular courses, principally because Algebra 30 is an entrance requirement for the large majority of colleges and faculties in the two universities and for some programs at SIAST. The Geometry-Trigonometry courses are also very popular. Although not required for general university entrance, completion of Geometry-Trigonometry 30 is required, with a few minor exceptions, for any mathematically-dependent program taken in the universities.

General Aim

The content, aims and underlying philosophies of the academic mathematics courses are outlined in the 1984 curriculum guide Algebra 10, 20 and 30 and Geometry-Trigonometry 10, 20 and 30. In a brief opening statement, the guide indicates that preparing students for a changing technological world is of primary importance and that the study of mathematics is an essential part of this preparation:

The predicted future of an increasingly complex technological society provides a major challenge for senior high school students. Increasingly, the necessity of a level of mathematical knowledge, beyond that required by today's intelligent consumer, is being brought to the attention of the public. If our graduates are to play a significant part in the technological world, it is important for both girls and boys to have a strong basic knowledge of mathematical concepts and be able to apply these to the world around them.

Thus, the academic mathematics program is intended to prepare students to understand the changing world around them and to succeed in their chosen careers. The curriculum guide states that students need to know mathematics "beyond that required by today's intelligent consumer." The extensive amount of content described in the curriculum guide implies that students need to have considerable more knowledge than the "intelligent consumer."

The guide indicates that the academic courses are "designed to provide the student with the necessary background in mathematics to enter post secondary educational institutions." Later, the guide becomes even more specific about the primary aim of these courses: To develop in students "sufficient mathematical background to enable them to study mathematics at a post secondary institution" [bold emphasis is mine]. The guide claims that the specified curriculum "enables students to meet the requirement for either mathematics and/or science or to prepare for a career in the professions" and, consequently, "in keeping with this academic requirement level, the major emphasis is on student acquisition of basic mathematical theory." The guide leaves no doubt that the academic courses are designed primarily to meet the needs of students planning to study more advanced mathematics when they leave secondary school.

Despite this emphasis on preparing students to enter mathematically-dependent careers, the guide argues that "the application of mathematical theory to 'real life' situations needs to be stressed if students are to see it as an essential life skill." Thus, the Academic Program is expected to prepare a minority of students for post-secondary technical and scientific courses, while ensuring that the large majority of students see the relevance of mathematics to "real life."

The Components and the Content

All the academic courses consist of a core and an elective component. The core component consists of subject-specific skills and concepts to be mastered by a majority of students. The elective component, which the guide confusingly states is mandatory, is "designed to provide a measure of flexibility in the course content to meet the needs of students and schools." It also enables the teacher "to spend time on topics which are not found in the core but which are of mutual interest to both the teacher and students." Seventy percent of the instructional time for each course, approximately 70 hours, is to be devoted to teaching the core content, while the remaining 30% or 30 hours is to be devoted to teaching the elective content.

Teachers are given the following choices to meet the requirements of the elective component:

- a. to introduce topics in these areas:

Grade 10. Consumer mathematics, History of mathematics, Linear programming, Transformational geometry, Mathematics and computer art, Construction in geometry.

Grade 11. Probability, Consumer mathematics, Absolute value, Matrices, Vectors, Transformational geometry, Projective geometry.

Grade 12. Statistics of grouped data, Introductory calculus, Matrices, Algebra and the computer, Normal curve and the binomial distribution, Topology, Geometric solids, Geometry and the computer.

b. to develop a unit at the local level to fit the needs determined by the teacher or school system.

c. to extend the subject matter under any of the core topics designed to give the student additional background beyond the core.

The guide indicates that the elective component does not need to be the same for all students in a course. For example, less capable students might study consumer mathematics, while the more capable students might study advanced topics in areas such as topology, vectors, probability or matrices. Teachers are also instructed to incorporate some of the core material from Geometry-Trigonometry 10 and 20 into the Geometry-Trigonometry 30 course, if these two courses are not offered in a school. This suggestion to teach content lifted from another course arises from a peculiarity of the academic program: neither Geometry-Trigonometry 10 nor Geometry-Trigonometry 20 is prerequisite for Geometry-Trigonometry 30; the principal reason being the inability or reluctance of smaller schools to offer the two prerequisite courses. The guide diplomatically refers to this program oddity, as "flexibility of course offering patterns."

Although the elective component appears to be a reasonable aspect of the academic program, several shortcomings are evident. First, the vagueness of choice "c" - to extend the subject matter of the core - allows a number of interpretations. For example, instead of providing students with enrichment, teachers might provide additional practice to master the core content rather than provide enrichment. Thus, it is quite acceptable to devote 100 hours instead of 70 hours of instruction to teach the core material and not introduce any additional content.

Second, other than the listing of topics under choice "a", teachers have been provided with a meager amount of information to help them develop the elective component. The guide mentions that objectives for the listed elected topics are under development; however, to the best of my knowledge, this development has never been completed.

Third, because the elective component can be so broadly interpreted, students who have different teachers are provided with quite differing instructional opportunities and mathematical backgrounds. For example, the amount of instructional time devoted to the four 1-credit university entrance courses, Algebra 10, 20, 30, and Geometry-Trigonometry 30, can theoretically range from 280 hours to 400 hours.

Moreover, Geometry-Trigonometry 10 and 20 and Calculus 30 provide some students with an additional 300 hours of mathematics instruction.¹ Thus, the mathematical preparation of students entering post-secondary institutions ranges anywhere from 280 hours to 700 hours of instruction. The wide range of background knowledge evident in introductory university mathematics courses can be partially explained by the variation in instructional hours provided to students in the secondary schools.

The bulk of the curriculum document presents a listing of the core objectives for each course. The specific learning outcomes of each course are organized under topic headings. The topics of Algebra 10 are polynomials and powers, solving equations (one unknown), linear inequalities, solving quadratic equations, real numbers, relations and functions, linear systems of equations (two unknowns). Algebra 20 consists of an extension of most of the Algebra 10 topics, principally by introducing more techniques, and the additional topics of ratio, proportion and variation, and statistics and probability. Algebra 30 extends some of the topics in the earlier courses and introduces the following topics: complex numbers; exponential and logarithmic functions; sequences, series and annuities; and permutations, combinations, and probability.

The geometry-trigonometry courses are organized by topics. Geometry-Trigonometry 10 includes the following topics: Points, lines, planes and angles; parallel lines, planes and polygons (both topics emphasize knowledge of terminology, simple constructions, and some basic postulates); congruent triangles; parallelograms; similar polygons (the preceding three topics involve the geometric theorems and proofs); right triangles (Pythagorean theorem and its simple applications); trigonometry (introduction to sine, cosine, tangent and their applications).

The following topics are studied in Geometry-Trigonometry 20: constructions (e.g., perpendicular to a line segment at a given point); the circle (postulates and theorems); areas of plane figures (e.g., triangle, trapezoid, etc.); areas and volumes of solids (e.g., prisms, cylinder, cone, sphere, etc.); analytic geometry (linear graphing, determining slope and describing lines as linear functions, distance from a line to a point, etc.); trigonometry (extension to other trigonometric terms and their applications).

Geometry-Trigonometry 30 is organized into two parts. In Part 1, the trigonometric ratios are covered more thoroughly than in Geometry-Trigonometry 10 and 20 and the following additional topics are introduced: trigonometric identities (reciprocal, quotient, Pythagorean, addition, double and half angle); trigonometric equations; graphing trigonometric functions (defining terms, identifying characteristics and sketching graphs); applications of trigonometry (law of sines and cosines). In Part 2, the focus is on analytic geometry: locus, circle, parabola, ellipse, hyperbola (knowledge of terminology and the general forms and applications); systems (e.g., linear-quadratic); general quadratic equation and conic sections.

The Alternative Mathematics Courses

The mathematics courses Mathematics 10, 20 and 30 were intended to be an alternative to the so-called academic mathematics courses. These three courses were expected to become less technical and less demanding than the academic courses, yet meet the general admission requirements for all post-secondary institutions in Saskatchewan, including the two universities. Unfortunately, for reasons to be presented later in this chapter, the legitimate goals of the alternative mathematics courses have never been met.

General Aim

The curriculum guide Mathematics 10, 20 and 30 published in 1977 clearly indicates the aims of the program: "The program is designed for those students who have post-secondary educational aspirations, but do not require the rigour of the present Algebra 30." The guide emphasises that the alternative program is "not designed to replace General Mathematics 10, 20, and 30 for pupils who desire a terminal course or who are low achievers." Kapoor (1974) provides a detailed list of general course objectives for the alternative mathematics program (p. 12):

The program should help the individual student to:

1. Improve mathematical skills by acquainting students with the mathematics involved in business, government and individual life.
2. Become familiar with concepts, acquire appreciation and understanding of principles and skills to the point of functional competence (formal proofs, etc., are de-emphasised).
3. Develop an interest and insight into mathematics through application in a variety of fields. (Problem solving will receive more attention than it has in the Dolciani program.)²
4. Appreciate the interplay between the development of mathematics and the development of our cultural heritage and to have a realistic but optimistic view regarding its future contributions to our way of life.
5. Develop a willingness and ability to apply mathematical concepts, skills and attitudes to situations outside the mathematics classroom.
6. Appreciate the importance of mathematics to vocations, professions, and many future studies, and to become aware of opportunities which are made available to individuals with various degrees of mathematical competence.
7. Fulfill his [sic] role in society in an intelligent manner, both as a producer and consumer.

8. Master those aspects of mathematics which are of value to the students rather than merely mastery for its own sake. Nevertheless, as in music, art and poetry, students should realize as part of their general education that mathematics has its artistic merits and has value for its own sake.

Table 1 indicates that all courses in the Alternative Program presently suffer from low enrolments; for every 1 student enrolled in the alternative program, about 14 students are enrolled in the academic program. Student enrolments in the alternative program have remained relatively low since the inception of the program in 1976, peaking in 1979 when approximately 16% of the grade 10 students were enrolled in Mathematics 10.

The Components and the Content

The bulk of the guide is devoted to listing the learning outcomes for each course and the corresponding textbook correlations. A more comprehensive document developed in 1976 containing the aims and objectives of the program as well as detailed instructional activities has never been published by the department.

Each course in the alternative program contains a core and an elective component. Teachers are instructed to devote approximately 80 percent of the instructional time in each course presenting the core content, leaving the remaining 20 percent for presenting the elective content.

A major strength of the alternative mathematics program is the integration rather than separation of mathematical topics. The program contains most of the topics studied in the academic mathematics courses, but the mastery of complex techniques and work with proofs are de-emphasised. Essentially, the alternative program is intended to be more "practical" than the academic program.

Mathematics 10 contains the following core topics: Business and consumer mathematics (e.g., simple and compound interest and their applications to credit and investment); exponents (working with powers and scientific notation); radicals and square root algorithms; fundamental operations on polynomials (e.g., operations on polynomials, factoring monomials and perfect trinomial squares); common types of graphs and descriptive statistics; graphing relations on the Cartesian plane; elementary properties of a linear function (e.g., slope-intercept form of a linear equation); graphing non-linear equations (e.g., the circle, ellipse, parabola); mensuration (e.g., working with SI linear and area units, and their relationships); Euclidean concepts (e.g., knowledge of terminology like point, line, etc., sum of angles of a triangle, relationships involving interior and exterior angles of polygons, Pythagorean relationship); ratio, proportion, and variation (solving proportion problems involving unit pricing, rates, etc.).

Mathematics 20 contains the following core topics: Business and consumer mathematics (emphasis on compound interest relationships and applications); radicals and exponents (e.g., simplifying expressions involving the square root of factors each raised to a power); fundamental operations on polynomials (e.g., operations on polynomials, solving linear systems and their applications); advanced interpretations of graphical information (e.g., interpolation and extrapolation); advanced properties of linear functions (e.g., equations of a line through two points); mensuration (e.g., working with SI area and volume units and relationships, area and circumference of circle, volume of selected solids); coordinate geometry (e.g., distance between two points, midpoint of a line segment); right angle trigonometry (the simple trigonometric ratios and applications, some identities); similar triangles and mechanical systems (direct variation, SI area and volume conversions, similar triangle problems, applying to mechanical systems); inverse and partial variation.

Mathematics 30 contains the following core topics: study of annuities (applications of geometric series to determine present and future values); the quadratic formula; sequences and series (applications of arithmetic and geometric series); parabolas (general form, sketching graphs); mensuration (mass, capacity and volume relationships), trigonometric functions (extended to cosecant, secant, cotangent, law of sines and cosines, basic identities).

In Mathematics 10, the following topics are electives: Owning an Automobile; Payroll; Insurance; Saving and Investing Money; Earning Money. In Mathematics 20, the following topics are electives: Second Degree Relations; Exponential Relations and Growth; Commission; Profit and Loss in the Retail Business; Municipal Taxation; Mathematics and Chance. In Mathematics 30, teachers are instructed to choose one topic from Income Tax, Stocks and Bonds, or Home Ownership, and one topic from More Conic Sections, Probability and Statistics, or Exponential Relations and Logarithms.

The topics of study chosen for this program are directly connected to those presented in the recommended textbooks. The developers of the program decided to create a program based on the available textbooks rather than create a program independent of any textbook series. As will be discussed later in this chapter, the lack of suitable textbooks has been one reason for the unpopularity of the alternative program.

Appraising the Secondary Mathematics Program

Members of past curriculum committees, teachers, mathematics educators, university faculty, parents, and secondary students have known for years that the secondary mathematics program needs to be revised considerably. When faced with major changes, Saskatchewan Education has typically responded by making only cosmetic

changes such as moving topics taught in one grade to another grade, recommending a new textbook for each grade, or including additional content to be taught. The assumptions underlying the secondary mathematics program are rarely examined or questioned. A major purpose of this chapter of the report will be to examine these assumptions and to raise questions about the purpose of secondary mathematics education. Until the purposes of secondary mathematics education are clarified, the selection and sequencing of content would be putting the cart before the horse.

Clearly the principal purpose for studying the type of mathematics contained in the current academic program is **to prepare students for the study of calculus offered in the post-secondary institutions, especially the universities.** The mathematical needs of students not intending to study calculus in post-secondary institutions are largely ignored or assumed to be equivalent to those of the more mathematically able and interested students. Even university-bound students not intending to study any further mathematics find that the entrance requirements of the universities force them to complete at least four courses in the academic mathematics program. Table 2 lists the secondary mathematics courses required for entrance into the major college and faculties in the two Saskatchewan universities. The entrance requirements for the other Western Canadian universities are included for comparison. Table 3 lists the high school mathematical requirements for selected programs offered by the Saskatchewan Institute of Applied Science and Technology [SIAST].

Table 2. Minimum Secondary School Mathematics Course Requirements for Entrance into the Two Saskatchewan Universities and into Selected Western Canadian Universities and into their Respective Major Faculties or Colleges and Introductory Mathematics Courses.

Colleges/Faculties	Requirements in each institution							
	UBC	SFU	UVic	UA	UC	UR	USask	UM
General entrance/ Arts	11	11	11	11	11	12+	12 ⁺	11
Agriculture	12	x	x	12	x	x	12 ^{**}	12
Engineering/ App. Sc.	12	12	12	12 ^{**}	12 ^{**}	12 ^{**}	12 ^{**}	12
Commerce/ Admin.	11	11 [*]	11	12	x	12 ^{**}	12 ^{**}	12
Dentistry	11	x	x	12	x	x	12 ^{**}	12
Education	11	11	11	11	11	12 ⁺	12 ^{**}	12
Medicine	11	x	x	12	x	x	12 ^{**}	12
Nursing	11	x	x	11 [*]	12	x	12 ^{**}	12
Pharmacy	12	x	x	12	x	x	12 ^{**}	12
Physical Education	11	x	11	11 [*]	11	12 ⁺	12 ^{**}	11
Science	12	11 [*]	12	12	12 [*]	12 ^{**}	12 ^{**}	12
Intro. calculus course	12	12	12	12	12	12 ^{**}	12 ^{**}	12
Introd. math course	11	11	11	12	12	12 ⁺	12 ^{**}	11

Source: 1988-89 University Calendars

B.C. universities: 11, 12 refer to Algebra 11, 12. New courses Mathematics 11, 12 are considered equivalent.

Alberta universities: 11, 12 refer to Mathematics 20, 30.

Saskatchewan universities: 11, 12 refer to Algebra 20, 30, unless specified otherwise.

Manitoba universities: 11, 12 refer to Mathematics 200, 300, unless specified otherwise.

* one or more grade 12 courses strongly recommended.

** two 12 courses required (Alta: Math 30, 31), Sask (Alg 30, Geo-Trig 30)

+ Math 30 acceptable. U of R (Arts, Educ. P.E., will accept Geo 20); U of Sask (only acceptable in Arts)

Table 3. Required and Preferred Secondary Mathematics Courses for Admittance into Selected Programs Offered by Saskatchewan Institute of Applied Science and Technology.

Type of Program	Length	Grade level ¹	Required/Preferred Courses	
			Alg. 30	Geo-Trig 30
Accountancy	2 years	12	P	
Administration	2 years	12	P	
Business Comp. Prog.	CBE			
Data Processing	2 years	12	P	
Food Services Admin.	2 years	12		
Hotel/Rest. Admin.	2 years	12		
Marketing	2 years	12		
Microcomp. Manage.	1 year	12	R	
Agric. Machine. Tech.	2 years	10		
Agric. Production Tech.	2 years			
Bio. Sci. & Animal Health	2 years	12		
Clerical and Related Prog.	CBE/33 wk	12/none		
Architech. Engin. Tech	2 years	12	R	R
Chemical Technician	38 wk	12	P	
Civil Engin. Tech.	2 years	12	R	R
CAD Technology	2 years	12	R	R
Comp. Engin. Tech.	2 years	12	R	R
Elec. Engin. Tech	2 years	12	R	R
Surveying Technology	2 years	12	R	R
Medical Lab Technology	1 year	12		
Nursing Assistant	1 year	11		
Nursing, Diploma	2 years	12		
Recreation Tech.	2 years	12		
Early Childhood Develop.	1/2 years	12		
Social Services Programs	1/4 years	11/12		
Trade Programs	8 wk/56 wk	8/11		

Source: 1988 University and Technical Institute Programs: A Career Planning Guide, Saskatchewan Education, January 1988.

¹ The minimum level of high school education. To graduate from high school, students must complete any grade 10 mathematics course and a mathematics or science course offered in grade 11 or grade 12. Beginning in 1988, students must complete a grade 11 mathematics course.

R refers to required; P refers to preferred.

An examination of Table 2 reveals that the two provincial universities are the only Western Canadian universities requiring a grade 12 academic mathematics course for general admission. All other universities require a completed grade 11 academic mathematics course. Moreover, students planning to enter specific programs in our provincial universities are required to complete more academic mathematics courses than students entering similar programs in other provinces. At the University of British Columbia, for example, a completed grade 11 course, Algebra 11, is a minimal requirement to enter the Faculties of Nursing, Commerce, Dentistry, and Physical Education, while the University of Saskatchewan requires two grade 12 courses, Algebra 30 and Geometry-Trigonometry 30, to enter these same Faculties (Colleges, to be exact). Even the entrance requirements in the University of Saskatchewan and the University of Regina differ. For example, the Faculty of Physical Education in Regina requires a completed Mathematics 30 for admission, while its Saskatoon counterpart requires both Algebra 30 and Geometry-Trigonometry 30.

Table 3 indicates that a few programs offered at SIAST require Algebra 30 or Geometry-Trigonometry 30. Considering the relatively low enrolment in these mathematically-dependent programs, the impact of the entrance requirements of SIAST on the mathematics curriculum is significantly less than that of the two universities.

The differences between university admission requirements in our province and the other Western provinces likely explains why student enrolment in the grade 12 academic mathematics courses offered in Saskatchewan are considerably higher than in other provinces. Table 4 compares the proportion of students in British Columbia, Alberta, Saskatchewan and Manitoba enrolled in either a Grade 11 or 12 course in mathematics course intended for students pursuing further studies at a post-secondary institution.

The similarity of the textbook series authorized by each province suggests that the provincial courses are similar in purpose, scope and general content. To determine the instructional hours of each course, I contacted by telephone a representative of the B.C. Ministry of Education and a representative of Alberta Education, both of whom have worked on the development of their respective provincial secondary mathematics courses. British Columbia recommends 120 hours for each secondary course (although the recent curriculum guide states 100 hours), while Alberta recommends 125 hours per course. However, the semestered schools in the provinces are experiencing difficulties in providing this amount of instructional hours. For example, I was told that some schools in British Columbia have been forced to provide two hour blocks of instruction including one hour over the lunch hour to comply with this time requirement. Whether secondary students in Alberta and British Columbia are provided with more course hours in mathematics than students in Saskatchewan cannot be directly answered by examining only the provincial recommendations.

Table 4. Number and Proportion of Students in British Columbia, Alberta, Saskatchewan and Manitoba Enrolled in either a Grade 11 or 12 Academic Mathematics Course.

Province	Grade 11			Grade 12		
	Course	Number	%	Course	Number	%
British Columbia	Alg. 11	21404	58	Alg. 12	11160	31
Alberta	Math 20	18650	58	Math 30	19677	54
				Math 31	5023	14
Saskatchewan	Alg. 20	11430	82	Alg. 30	10268	72
				G-T 30	7101	50
Manitoba	Math 200	7046	48	Math 300	6843	41

Sources: Public Schools Data, 1989, Ministry of Education, Province of British Columbia; 82nd Annual Report, 1986-87, Alberta Education; Saskatchewan Education Circular, 1988, Evaluation and Student Services; Correspondence with W. Wyatt, Curriculum Development and Implementation Branch, Manitoba Education.

Note 1: These figures were taken from the annual reports provided to me by each province. Unfortunately, not all the reports are for the current year. However, while the number of students enrolled in these courses varies from year to year, the proportions remain relatively constant. Consequently, I believe these figures to be reliable for the purposes of comparison.

Note 2: The proportion of students in a grade taking a mathematics course was calculated by dividing the student enrolment in a course by the student enrolment for a grade. This calculation does not exclude students enrolled in another grade (for example, grade 10 and 12 students taking a grade 11 mathematics course). Consequently, each proportion is an overestimate. I have assumed that the proportion of students taking a mathematics course in a year different than their grade level is constant from province to province.

Compared to the other provinces, a disproportionate number of students in Saskatchewan take the courses primarily intended for those students wishing to pursue a further studies in mathematics in a post-secondary setting.³ Why do so many more students in Saskatchewan than students in other provinces take the academic mathematics courses? Because the academic mathematics courses are very similar from province to province, the popularity cannot be the result of a more appealing curriculum. The popularity of these courses is almost entirely related to the desire of students to meet university admission requirements.

Should we expect this number of students to complete the academic mathematics courses? No evidence exists to indicate that the universities, businesses, industries, and government agencies require this number of students with the type and level of preparation developed in the academic mathematics program. At the risk of stating the obvious, the large majority of these students will not study mathematics beyond that taught in high school. Based on information provided by Saskatchewan Education and the University of Saskatchewan and on studies of school leavers, it is estimated that less than 20 percent of students who complete grade 8 will continue on to complete any first-year university mathematics course and less than 5 percent will complete a mathematics course beyond an introductory one.^{4,5}

Few high school students will need or desire the mathematics and science courses offered at the universities and, unless there is either a dramatic increase in demand for qualified technicians and scientists or the job entry requirements of the non-technical professions change, the number of students requiring courses in advanced mathematics is not expected to increase significantly in the future. Yet over 7000 students a year graduate annually with credit in both Algebra 30 and Geometry-Trigonometry 30. It seems unreasonable to continue to base the high school mathematics program mainly on the needs of a relatively small group of students, namely those preparing for scientific and technical careers.

The enrolment in the academic mathematics program is also influenced by "credentialing." In order to reduce the number of applicants for a job or a post-secondary program, employers and institutions sometimes select applicants who have successfully completed the academic courses, although only a limited knowledge of mathematics is essential for success.⁶ The Issues and Options report on accessibility recently completed by the University of Saskatchewan (1989) examines the issue of credentialing:

... a high production of university graduates in the 1970s 'led to credential inflation as these graduates took jobs where degrees were never previously needed. As a result, degrees are not needed now so much to advance in socio-economic condition as to remain level. **Post-secondary education is not a bonus, not a luxury, not a passport to an elite, but**

a practical necessity and a prerequisite for the well-being of literally hundreds of thousands of people in Saskatchewan alone. (p. 14).

The report adds: "A large majority of Canadian high school students, approximately 86% expect that their education will not end with their high school diploma. Fully one-half of high school students expect to finish university; not because they *want to do so*, but because they believe they *need to do so*" (p. 14). Unfortunately, the report claims, this number of graduates far exceeds the available university places in Canada.

Past attempts to reform the secondary mathematics program in Saskatchewan so that it better meets the needs of all students have not been successful, a good example being the Mathematics 10, 20 and 30 program. As has been mentioned earlier in the review of this program, the original intent was to develop a "middle-of-the-road" secondary mathematics program whose completion would allow students to enter post-secondary institutions in areas where a knowledge of advanced mathematics is unneeded.

The program was expected to attract the large majority of high school students. Once this goal was accomplished, the academic courses were to be revised to be more suited to students interested in preparing for careers where further serious study of mathematics is required. Thus, the mathematical needs of secondary students who differed in mathematical interest and ability could be met by one of two programs, either the Mathematics 10, 20, 30 program or the Algebra 10, 20, 30 and Geometry-Trigonometry 30 program.

The enrolment in the Mathematics 10, 20, 30 program has never lived up to the expectations of the original Division IV Mathematics Curriculum Committee. Prior to its implementation, about 60 to 70 percent of secondary school students were expected to enrol in the Mathematics 10, 20, 30 program.⁷ The Division IV Mathematics Curriculum committee believed that the existence of the alternative program would allow the academic program to be eventually enriched and strengthened so that only those students who were mathematically inclined (approximately 20 to 30%) would elect the academic program. The remaining 10 to 20% of secondary students would enrol in a general mathematics program.

The enrolment figures for the Alternative Program are a sign of its failure. In 1987-88, only about 5 percent of grade 12 students enrolled in Mathematics 30, the final course of the program. In recent years, fewer students take Mathematics 30 than Calculus 30, an elective mathematics course.

For several reasons, the Mathematics 10, 20, 30 program has never become popular. First, most colleges in the universities did not change their entrance requirements as had been expected. Even non-technical colleges like the College of Physical Education

at the University of Saskatchewan insist that entering students complete Algebra 30 and Geometry-Trigonometry 30 rather than Mathematics 30.⁸ The College of Education at the University of Saskatchewan did allow students enrolled in the Elementary program to enter with credit in Mathematics 30; however, these entrants found that Algebra 30 and Geometry-Trigonometry 30 were prerequisite courses for Mathematics 102.6, a required course in the first year of the Elementary program. Consequently, they found themselves in an absurd "Catch 22" situation of meeting the admission standards for the College, but not satisfying the mathematics requirements for one of the first year courses. Needless to say, students complained bitterly. The College of Education now requires Algebra 30 for admission.⁹

High school students planning to enter university quickly realized that, despite the claim stated in the curriculum guide, the Alternative Mathematics program was not a university entrance program. The moribund state of the Mathematics 10, 20 and 30 courses is primarily due to the lack acceptance by the post-secondary institutions, principally the University of Saskatchewan.

Second, because the Mathematics 10, 20, 30 program was based initially on the currently available textbook series, the program has never been placed on a solid philosophical footing. Without clearly stated goals, the type and amount of content to be included in the program cannot be judiciously selected. Consequently, a large amount of the mathematics content is inappropriate. For example, Mathematics 30 includes a detailed study of annuities where the formulas for the present value of an annuity are presented as special cases of a geometric series. This topic of geometric series is treated more abstractly in the Mathematics 30 course than in the supposedly more rigorous Algebra 30 course. Teachers who teach or who have taught the Mathematics 10, 20 and 30 courses have never clearly understood the purposes of the program. Not surprisingly, most schools continue to offer the academic courses whose purposes are clearly understood rather than offer a seemingly confused program.

Third, smaller schools do not have the resources to offer the range of secondary mathematics courses authorized by Saskatchewan Education. About half the secondary schools in Saskatchewan have an enrolment less than 150 students. These smaller schools cannot offer the same number of electives offered in larger schools. A small school can offer on average about 33 credit units, while larger schools (over 900 enrolment) offer on average about 110 credit units. Because students in the smaller schools as well as their parents tend to demand "university entrance courses," smaller schools quite understandably choose to offer the academic courses instead of the seemingly terminal Mathematics 10, 20, 30 program or the General Mathematics program. Table 5, taken from a study of program delivery in the small rural high schools (Ryan, Sackney, & Birnie, 1981), indicates how the university indirectly forces schools to offer only academic mathematics courses.¹⁰ The data in Table 5 clearly

indicates that smaller rural schools do not consider the alternative mathematics program as an alternative.

Table 5. The Types and Numbers of Secondary Mathematics Courses Offered in 1981 by Small Rural Schools in Saskatchewan.

Course	<u>Grade 10</u> (N = 91)		<u>Grade 11</u> (N = 90)		<u>Grade 12</u> (N = 89)	
	Teacher.....	Other	Teacher	Other	Teacher	Other
	assigned.....	means	assigned	means	assigned	means
Modern Algebra	89.....	0	87	0	86	0
Geometry	79.....	1	66	0	82	1
Mathematics	0.....	0	0	0	0	0
General Math	17.....	5	17	3	9	1

Ryan et al. report that a minimally adequate program must serve two primary functions: (1) to prepare students for life, and (2) to qualify them to enter any post-secondary educational institution (p. 72). The researchers argue that the academic courses are not necessarily relevant to all high school students, yet small schools must inevitably offer them:

The reality of the Saskatchewan situation is that both students and parents make program choices that will keep the occupational doors open as long as possible; the preferred program choice is the academic curriculum. One of the reasons that schools offer essentially an academic program is that it leads to future careers which are most rewarded by our society.

More recently schools now consider it necessary to offer Calculus 30, a local elective. They believe that this course gives their students a considerable advantage over other students who did not study Calculus 30.¹¹ Smaller, rural schools are now under considerable pressure to offer Calculus 30 and the Alternative Program can be expected to virtually disappear from all schools but the larger collegiates.¹²

Footnotes

- ¹ Some larger schools ignore provincial policy and require students to complete both Geometry-Trigonometry 10 and 20 before Geometry-Trigonometry 30 can be taken. A few schools require students to complete all six courses before taking Calculus 30, a local but increasingly popular elective.
- ² The academic program has been referred to as the "Dolciani program" because the most popular authorized text for the program at the time of its creation was Mary Dolciani.
- ³ The Second International Mathematics Study (Robitaille, O'Shea & Dirks, 1982, p. 4) reports that the figure for Ontario is about 40 percent. The number of students in other countries who have taken the highest level of mathematics available in their secondary school system required to specialize in the study of mathematics or science at the post-secondary level are reported as follows: U.S.A. (12%), England (30%), France (28%), Japan (14%), New Zealand (67%), and Scotland (4%).
- ⁴ This figure is based on the estimate that 30% of grade 8 students do not complete grade 12 (Cipywnyk, Pawlovich, & Randhawa, 1983) and about 40% of grade 12 students enter one of the two Saskatchewan universities immediately after graduation (Saskatchewan Education, 1987). Recent studies suggest that the 30% dropout rate is likely an underestimate.
- ⁵ The report Everybody Counts claims that in the U.S.A. about roughly half of the student body annually from grade 9 drop mathematics as a subject of study. Consequently, for every 5000 American ninth graders, about 10 will graduate with a Bachelor of Science, about 4 will graduate with a Masters of Science and only 1 will graduate with a Ph.D. Students in most American high schools are not required to study mathematics after grade 9.
- ⁶ As an example: I was informed by the representative from Alberta Education that Math 30, the Alberta grade 12 academic course, is a requirement for a post-secondary Travel Agent program.
- ⁷ I have been unable to find any reference to a figure in any document published by Saskatchewan Education. However, several members of the 1974-1978 Division IV Mathematics Curriculum Committee have told me that they believe that the alternative program was expected to attract from 60 to 70 percent of secondary school students. A review of the literature yielded a little more information. Kapoor (1974) states that the alternative program "meets the needs of the majority (65-70%) of the students" (p. 16). Giesbrecht (1977) cites an unpublished paper in 1973 by Ramsay, Rumpel, and Simington: "[the alternative program] would meet the needs of the average student, and it was envisioned that ideally the majority (65 to 70 percent) of students would enrol in this program" (p. 86).

- ⁸ The recursive chain of reasoning goes something like this: Physical Education graduates need to know some mathematical statistics, especially hypothesis testing. Mathematics 245.3 is the appropriate course: the prerequisite being a university course in mathematics. All university courses at the University of Saskatchewan require Algebra 30 and Geometry-Trigonometry 30.
- ⁹ The Mathematics Department of the University of Saskatchewan has argued for years that the College of Education should change its entrance requirements for the Elementary Teacher Education Program from Algebra 30 to both Algebra 30 and Geometry-Trigonometry 30, the prerequisites for Mathematics 102.6. A committee is currently examining the need for the prerequisites and the content of Mathematics 102.6
- ¹⁰ Small high schools are defined by Saskatchewan Education as having less than 20 students enrolled in any one of grades 10, 11 or 12.
- ¹¹ The perception of smaller schools that their students are disadvantaged by not taking Calculus 30 is an accurate one. A study of performance in Mathematics 175.6 at the University of Saskatchewan (Scott and Stephenson, 1985) found a statistically significant difference in average grades in Math 175.6 between students who did and did not take Calculus 30 even when adjusted for differences in achievement in the prerequisite courses, Algebra 30 and Geo-Trig 30. Because the difference was at least one grade level higher for students taking Calculus 30, the authors recommended that "high school students who contemplate taking Math 175.6 should be encouraged to take Calculus 30 if it **is available**" [bold emphasis is mine].
- ¹² Most provinces are experiencing rapid growth in enrolment in high school calculus courses. As in Saskatchewan, concerns about equity in educational opportunity have been raised. For example, in British Columbia, the AP calculus course, which is not recognized by the Ministry of Education and hence no enrolment statistics are available, has experienced a rapid growth in student enrolment, especially in the Lower Mainland. I was told by a Ministry official that some high schools attract students because they offer the AP calculus course, which is perceived as good preparation for the first year university calculus course because, as he expressed it, "The university course is essentially a repeat of the same course work as the AP calculus."

CHAPTER 16

REVISING THE SECONDARY MATHEMATICS PROGRAM

In this chapter, changes to the present Saskatchewan secondary mathematics program will be proposed. These proposals are based on the literature review, arguments, and discussion presented in Part 1, in the discussion of the elementary and middle years program presented in Chapters 5 through 14 and in the review of the current secondary mathematics program presented in Chapter 15.

I avoided the temptation to specify in mandarin fashion a detailed list of learning outcomes for each course in the secondary mathematics program. Instead, I will identify the factors that must be considered before any serious revision to the program can begin. I will also discuss various ways to organize and structure the secondary mathematics programs to meet the needs of more students than are being presently met by the current secondary mathematics program.

The following tasks need to be carried out by Saskatchewan Education to ensure a successful revision of the secondary mathematics program:

1. Establishing realistic goals for all courses in the program.
2. Developing a flexible program structure.
3. Selecting and sequencing the mathematics content and learning outcomes for all courses.
4. Identifying and developing appropriate instructional materials.
5. Providing for the preservice needs of prospective secondary mathematics teachers and the inservice needs of practising teachers.
6. Evaluating the implementation of the revised program.

This report will consider only the first three tasks. The remaining tasks are more properly the responsibility of Saskatchewan Education as represented by the curriculum writers to be hired eventually, the K-12 Mathematics Advisory Committee and other educational support groups. Task 3, the selecting and sequencing of content and learning outcomes, can only be attempted after the completion of the first two

tasks. Consequently, a detailed discussion of content change will not be included in this report.

Establishing Realistic Goals

As argued in Part 1 and in the earlier part of this chapter, the purposes of the secondary mathematics program have to be clearly specified and agreed upon before any major revisions can begin. To meet the needs of students whose careers and lives will take place in the 21st century, the secondary mathematics program must be designed to accomplish the following general goals:

1. Prepare students to live in a society becoming increasingly dependent upon technology.
2. Prepare students who are numerate and, in particular, can understand arguments based on quantitative information.
3. Prepare liberally-educated students who understand the nature of mathematics and the way mathematics knowledge is created and the role of mathematics in a modern society.
4. Prepare students for careers that require some post-secondary training.

Preparing Students for a Technological Age

Contrary to the claim that a knowledge of mathematics is necessary for living in the imminent technological and information age, and regardless of the obvious dependence of technology on the progress of mathematical knowledge, "the overwhelming majority in a modern society can and do live quite well **without doing nearly any mathematics**" (Kietel, 1988, p. 7). If the rapid expansion of knowledge about mathematics and its uses in science and technology is to continue, the preparation of a professional minority is an important consideration, but not necessarily in designing a high school mathematics program. Advocates of mathematics education need to be more realistic and recognize that the increasing **mathematization** of society leads to an **increasing demathematization** of its members (Keitel, p. 7). The mathematics underlying a technology is rarely apparent to those who use it.¹

In revising the mathematics curriculum, we need to ensure that students are not learning mathematical skills that have been or are being rapidly obsolesced. Moreover students should learn the appropriate use of technology such as calculators and computers. The emphasis here should be on applying mathematical knowledge to solve interesting or realistic problems.

Preparing Numerate Students

The study of mathematics should ensure at the very least that students become sufficiently numerate to function effectively in the world outside of school: that is, knowing how to use mathematical ideas to carry out routine tasks, to solve realistic problems, and to interpret commonly seen quantitative statements (Hope, 1987b). Although the development of numeracy is not the exclusive responsibility of mathematics teachers, clearly the mathematics program should be largely responsible for ensuring that young adults leave school with the ability to do the following types of tasks independently and with a minimum of intellectual effort:

- read and interpret quantitative information found in newspapers, popular magazines, government, political and business publications and to evaluate arguments based on such information.
- understand sufficient mathematics necessary for wise consuming, working in most non-technical work settings, and working with commonly seen technological devices.

The current mathematics program with its overemphasis on abstract ideas and mathematical procedures unrelated to everyday life does not help students become sufficiently numerate. For example, although studies consistently demonstrate that students and adults have difficulty understanding percentages, this mathematical topic is rarely studied after grade 8. A revised program should contain a greater number of more practical topics than found in the present secondary mathematics program.

Preparing Liberally Educated Students

As argued in Part 1 of this report, the content of a revised mathematics program should not be exclusively "practical." The program should not be merely a series of courses on consumer and career mathematics. The program should help students understand the nature of mathematics and the way mathematics knowledge is created as well as develop an appreciation for the role mathematics plays in a modern society.

Students should learn some "advanced" mathematical ideas that have few applications outside of mathematics. The study of number theory and deductive proof in geometry, for example, serves a similar role in the development of an educated person as the study of Shakespearean sonnets. The real question is this: How many and what advanced notions should be included in a revised program?

Students should also learn the roots of mathematical knowledge, how it was developed and who were its major contributors. All students should leave school

knowing some rudimentary history of our number system, for example, if they are to be considered to be liberally educated.

Preparing Students for Careers

For the sake of equity, all students must be given adequate preparation for a variety of possible careers as well as any further education they may decide to elect. But how can this be done when students are not equally advantaged. They differ significantly in abilities, interests, and in many other important ways. To assume that all students are equally capable in learning mathematics is foolish and irresponsible.

The goal of preparing all students for indeterminate careers is the major reason why the secondary mathematics program is so difficult to revise. A revised secondary mathematics program has to be structured carefully to ensure that the career needs of all students can be met. As has been argued previously, the academic mathematics program does not meet the needs of the majority of students not intending or desiring to study post-secondary mathematics. Perhaps, as many university professors argue, it does not even meet even the needs of those students who do intend to study post-secondary mathematics.

Although proportionally fewer Saskatchewan people enter post-secondary education than in other provinces, the authors of Issues and Options believe that a reasonable goal for Saskatchewan is to provide a post-secondary education for approximately 60% of its people with at least 30% obtaining a university degree. Whether universities object to the changes or not, post-secondary education is rapidly evolving from a community of the academically elite into an extension of the school system. Consequently, the influence of the universities on secondary curricula is currently being questioned.

Although the universities have the authority to set entrance standards, they need to work cooperatively with Saskatchewan Education to ensure that these standards are reasonable. In the meantime, curriculum developers cannot ignore the present entrance requirements established by the post-secondary institutions. Program revision will not be successful unless the course sequence is recognized as being suitable preparation for entry into post-secondary programs.

A major task for Saskatchewan Education will be to develop a program structure considered acceptable by all educational stakeholders including the universities; a structure that takes into account the post-secondary aspirations of students, the needs of the universities and other post-secondary institutions, the individual differences of students, and the instructional capabilities of smaller secondary schools. Until a more suitable structure is developed, however, serious curriculum reform cannot begin. In

the next section, some possible structures along with their advantages and disadvantages will be discussed.

Developing a Program Structure

The purpose of this report is not to identify and recommend a single structure but to present some possible alternatives to the current secondary program. Three structures will be discussed:

1. A modification of the current dual university entrance program structure.
2. The structures proposed for other comparable provinces, in particular, British Columbia and Alberta.
3. A single program structure with multiple exits.

Structure 1: A Modification of the Current Structure

A simpler task than creating an entirely new and unfamiliar structure would be to modify the present dual university entrance mathematics program to ensure that it becomes more effective in meeting the needs of all students. Program A consisting of three courses, Mathematics 10A, 11A, and 12A, would be a revision of the Alternative Mathematics Program and be designed for those students intending to enter a post-secondary institution, but not intending to enter a program requiring a university level calculus course. Program B consisting of four courses, Mathematics 10B, 11B, 12B1 and 12B2, would be a revision of the Academic Program and be designed for students who have the interest and ability to study the mathematics needed for scientific and technical careers.² The large majority of students would be expected to complete Program A. A small minority of students would be expected to complete a revised General Mathematics or Life-Skill Mathematics Program offered as local electives.

The following modifications must be made to ensure the successful implementation of a dual program structure:

1. Completion of either Math 12A, 12B1 or 12B2 would satisfy the entrance requirements for the two universities.
2. Program B would be designed for students preparing for programs and careers requiring a great deal of mathematical background and, consequently, completion of 12B2 would be required as prerequisite for entrance into these programs.

3. Although many topics in Program A and B would be similar, the pace of presentation, depth and breadth would differ. At least one course would be designed to allow students in Program A to transfer to Program B. Students in Program A would likely not need such a transfer course.

4. Both programs would be available to any secondary student in Saskatchewan, regardless of geographical location and the resources of the school.

5. Electives in Program B would be developed by Saskatchewan Education to ensure that mathematically interested students had an opportunity to study as much mathematics as they desire. While some Colleges and programs might strongly recommend certain elective courses, none would be required. Electives would also be available for students in Program A who desire further study in practical mathematics, Mathematics for Careers, for example.

6. The development of Program B would begin after consultations with the post-secondary institutions, especially the universities. All post-secondary programs having a mathematics requirement would be analyzed to determine the skills and understandings needed to be included in Program B to prepare students for these programs.

7. The core courses in Program B would be integrated and contain many different mathematical topics: algebra, geometry, trigonometry, probability, statistics, number theory, discrete mathematics, complex numbers, and some of the underlying concepts of calculus - limit theory, for example. Single theme courses such as the current Algebra and Geometry-Trigonometry would not form the core.³

This dual university entrance program structure is based on the assumption that a single secondary school mathematics program cannot adequately prepare a minority of students for mathematically demanding careers while simultaneously preparing the majority for careers requiring only a modest knowledge of mathematics background. Individual differences in ability and interest are more apparent in mathematics than in almost all other subject areas and, therefore, a single program usually cannot meet the needs of all students.

On the one hand, there are a small minority of 12- and 13-year-old students who learn mathematics so rapidly that they can complete the entire high school courses in Algebra and Geometry in only one summer school session.⁴ On the other hand, there are large numbers of students who find it difficult to absorb mathematical abstrac-

tions, despite considerable attempts to slow down the rate of presentation and to modify the instructional approach.

Most large secondary schools in Saskatchewan currently try to accommodate for individual differences in learning mathematics by offering differentiated curricula for classes of students grouped by ability and past performance. In some larger high schools, at every grade level, as many as three versions masquerade as an academic mathematics course: e.g., Algebra 30A, Algebra 30B, Algebra 30C. For several reasons, the practice of arranging a course into various versions for different types of students is unsatisfactory.

First, students receive the same credit for each academic mathematics course, regardless of how much the content and pace of instruction have been tailored to their needs and abilities. In the University of Saskatchewan, quotas on first-year student enrolment unfortunately now exist. Students who choose to take a more demanding version of a course such as Algebra 30 (Enriched) often receive lower grades than students in the regular version of the same course and, consequently, can find themselves at a disadvantage when seeking entrance into the University because entrance depends solely on high school grades.⁵

Second, the offering of slightly different or slower paced versions of courses whose intended purpose is to prepare students for entering mathematically demanding careers seems an unsatisfactory way to meet the needs of the large majority of students who will never take advantage of this preparation. If students are expected to graduate as educated persons, we should ensure that they have been at least introduced to some of the cornerstones of mathematics, the basic elements of trigonometry, algebra, logic, Euclidean geometry and so on. But is it reasonable to expect all students to master the myriad of technical details needed to prepare students for a post-secondary calculus course?

As has been argued earlier in this chapter, the current academic mathematics program enjoys a high enrolment for only one reason: It is a university entrance requirement. But access to the universities is determined by entrance requirements that can be modified. Why should all students who intend to acquire a university education be compelled to meet the requirements set primarily by the mathematics, science and engineering departments when they have no intention of pursuing a mathematically demanding career?

The creation of dual university entrance programs to differentiate between students with different needs and capabilities appears to be consistent with the recommendations of the influential Standards committee of the National Council of Teachers of Mathematics (1989). They proposed that all American students should be required to take three years of a core curriculum differentiated in terms of depth and breadth of

treatment and the nature of applications. They are emphatic that a core curriculum "does **not** imply that students of all performance levels must be taught in the same classroom, and it does **not** imply that the content presentation for all students must be the same" (p. 130). Furthermore, they differentiate between the **content topic** being the same and the **content** being the same. That is, although all students should learn the same curricular topics, the treatment should vary according to the performance levels of the students being taught.

The NCTM Standards committee proposes that the typical "General" or "Remedial" mathematics courses for less able or disadvantaged students where essentially sixth grade mathematics is repeated over and over again should be replaced by a central core of mathematical ideas: ideas about mathematical representation, mathematical processing, mathematical problem solving, and mathematical thinking. If courses designated as General Math do not address the goals of the associated core curriculum, they are no longer acceptable.

Obviously, if two university entrance programs are to be developed and successfully implemented, the universities and other post-secondary institutions will have to be consulted. Under the University Act, the universities have the authority to determine the standards and requirements for entry into all colleges and faculties. But Saskatchewan Education and all the post-secondary institutions must collaborate to ensure that entrance requirements and standards are realistic and in keeping with the spirit of the Core Curriculum: The more conditions placed on university entrance, the less opportunity students have to complete all aspects of Core.

The Core Curriculum Policy Advisory Committee understood the importance of improving communications between all educational stakeholders in the province. The following quote taken from the report Program Policy Proposals published in 1986 emphasises this point:

Program policy decisions cannot be made in isolation from the post-secondary education community. It is essential in the 1980's for students to understand that education does not begin and end in the public school. Decisions about program, therefore, must be made in full knowledge of the needs and expectations of universities and technical training institutions. On the other hand, **post-secondary institutions should not control what is taught in secondary schools** [bold emphasis is mine], and ongoing articulation between elementary-middle-secondary education and post-secondary education is necessary. Prior to adopting a revised secondary education policy, policy-makers must explore the implications of their proposals and assure parents and students that they will not be caught between competing educational policies. (p. 36)

The report prepared by the British Columbia Royal Commission on Education (1988) expresses similar concerns as Directions about the enormous influence of university entrance requirements on secondary school programs. The report presents several examples to support the claim that "with little if any consultation with the public school system, universities make **arbitrary decisions** [bold emphasis is mine] about what will, or will not, be accepted as meeting university entrance requirements" (p. 106).⁶ The report argues that dictates by the university about entrance requirements, no matter how well-intentioned "are simply not acceptable, especially since much-needed reforms in the secondary schools are thwarted by them. The proper role of the universities must be to offer advice concerning the most appropriate content for the curriculum of the secondary school, but not to dictate that content or to take unilateral action which might adversely affect school programs" (p. 106).

The Articulation Committee formed by Saskatchewan Education provides our province with an opportunity to begin the badly neglected process of mutual consultation between secondary schools and post-secondary institutions. Unless both programs I have described are considered acceptable for general university entrance, students will not enrol in Program A and the resulting increased enrolment in Program B will force teachers to deviate from the originally intended goals.

Designing Program A. Each course in Program A should consist of a variety of topics rather than be an indepth treatment of only algebra, geometry or trigonometry. Program A should attempt to provide students with an integrated view of mathematics. To be consistent with the goals of the elementary and middle years program, the secondary program should plan for student success and, in general, be based on a minimal number of mathematical ideas seen in a variety of applications and contexts.

This program is not intended to be devoid of mathematical ideas. Rather, it can be based on the broad strands suggested in the NCTM Standards report including the traditional areas of algebra, geometry, and trigonometry, and the newer topics of statistics, probability and discrete mathematics. In addition to learning the foundations of mathematics, students should also continue to build on their learning acquired during their middle years and study what is commonly viewed as "advanced arithmetic" including ratio and proportion, percentages, estimation, mental calculation, data organization and analysis, measurement, and so forth. The aim would be to produce numerate citizens who can readily apply this knowledge to solve everyday problems.

Designing Program B. Although the specific learning outcomes for each core and elective course should be developed collaboratively by Saskatchewan Education and the post-secondary institutions, Program B should be based on the following general aims:

1. To develop the ability to formulate in mathematical terms problems that exist in a variety of situations and to solve them in logically correct and systematic ways.
2. To develop an appreciation of the importance of mathematics in society and how it contributes to our understanding of phenomena.
3. To develop an understanding of the fundamental concepts, laws, principles and relationships in mathematics, especially those that underlay calculus.
4. To develop insight and the ability to produce a sound mathematical argument.
5. To develop a facility with advanced mathematical techniques and the knowledge to decide when to use technology and how to use it.
6. To develop the ability to learn mathematics independently with only minimal guidance from others.
7. To develop a willingness to tackle challenging problems and to endure any associated frustration.
8. To develop the ability to discuss and exchange mathematical ideas with others and to accept their valid criticism.

One important goal of the program should be to help students develop an integrated view of mathematics, especially how one field relates to another. Topics need to be carefully selected with this goal in mind.

Although students in Program B should learn a number of mathematical techniques, these techniques should be learned either as preparation for learning other important but advanced mathematical techniques or as tools for solving problems in related fields. Consequently, each course should contain a number of illustrative applications drawn from disciplines such as physics, economics, social sciences, business & industry, engineering, and so forth.

Students in Program B as well as in Program A would be encouraged to use and have access to a variety of technological aids to reduce the drudgery of carrying out long and cumbersome mathematical procedures. Students would learn how to use scientific, financial, and statistical calculators and computer software to process and analyze quantitative information. The emphasis would be on using technology to assist in the solving of problems. Students would be expected to use the technology

intelligently and to be able to interpret the machine-derived data. There would be an opportunity for students to use symbolic manipulators such as muMATH.

The use of technology in mathematics will be promoted only if the following conditions can be met: (1) specific learning outcomes are incorporated into the program; (2) facility with technology is evaluated; and, (3) the technology is available to all.

Students should be taught how to learn mathematics on their own and be given adequate practice to acquire this independence. This skill is especially important for students intending to study mathematics, science and engineering at the universities where the rapid pace of lectures necessitates much outside reading. Students would have access to a variety of textual materials and there would be an emphasis on learning how to read mathematical presentations.

A critical first step in the design of Program B would be for Saskatchewan Education to ask every post-secondary institution to examine their program offerings in detail with the purpose of identifying the mathematical concepts, skills and understandings needed to succeed in each program. The task would be to identify the essential rather than merely the desirable. The post-secondary institutions must go beyond the traditional method of selecting the most demanding courses as entrance requirements, merely to serve as convenient screening devices.⁷

Only when the post-secondary institutions have specified the types of understandings and skills that they assume will be acquired by secondary students can the content of the core courses and electives in both programs - Program B, in particular - be determined. This consultative process might also lead to changes in some post-secondary courses as well.

Whatever learning outcomes are developed for each core course in Program B, it is recommended that little deviation from the prescribed content be permitted. In the current academic program, the elective component in each course allows for only a superficial treatment for the topics that are suggested in the curriculum guide: probability, statistics, transformational geometry, linear programming, and so on. I expect that most teachers have ignored these topics anyway because they seem tangential to the main body of each course of the program. Consequently, I believe that it is preferable to develop elective courses that provide sufficient depth for the capable student rather than incorporate an elective component into each of the four core courses.

The elective courses should include the mathematical content considered fundamental to fields of mathematics and related disciplines. They should also contain mathematical ideas thought to contribute to the development of mathematical thinking or

considered to promote an appreciation of the traditional foundations of mathematics. Any of the following topics could be developed into a sound elective course:

1. Euclidean Geometry
2. Inferential Statistics
3. Probability
4. Discrete Mathematics
5. Transformational Geometry
6. Number Theory
7. An Introduction to the History of Mathematics
8. Analysis
9. Projective Geometry
10. Linear Algebra and Linear Programming
11. Calculus

Each of the above topics is sufficiently broad to be divided into other elective topics. For example, the broad topic Discrete Mathematics can be separated into any number of finer elements including combinatorics, logic, set theory, Boolean algebra, matrices and determinants, and vector analysis. Perhaps, some elective topics should be developed into 0.5 credit courses (approximately 50 class hours) instead of the more traditional 1 credit courses.⁸

Barriers to the development of a dual program. The barriers to the development of the proposed dual program are the same ones that continue to hinder the delivery of the two current secondary mathematics programs: (1) acceptance of Program A for general university entrance; (2) the differing instructional resources of secondary schools; (3) the selection of students for each program; (4) the number of instructional hours provided for Program B.

1. Acceptance of Program A. The acceptance of Program A for general university entrance is essential if the dual program is to be workable.

Saskatchewan Education and the post-secondary institutions, especially the universities, must cooperate and carefully review the admission requirements of all programs.

Another potential stumbling block is the entrance standards for other provincial universities, who may not recognize program A for admission. While the entrance standards of out-of-province post-secondary institutions should not be ignored, they surely cannot be used to determine the Saskatchewan mathematics curriculum. Otherwise, we should simply adopt the curricula developed in our neighbouring provinces and disband further curriculum development in Saskatchewan.

2. Instructional resources. Larger high schools usually have the resources and student enrolment to offer distinct secondary mathematics programs by using the traditional means of assigning a group of students to a teacher. However, most smaller schools have neither sufficient resources to offer nor the number of students to justify several alternative programs. Because the academic programs are presently required for university entrance, rural schools currently offer these courses to ensure that their students have at least an opportunity, whether taken or not, to enter a university.

The data in Table 1 provides information about secondary schools and their enrolments. Fifty-five per cent of the students in the province attend large city secondary schools, with an average enrolment of 498 students in 1984. Forty-five per cent of our students attend rural high schools, with an average enrolment of sixty-four students. The average enrolment for all secondary schools in Saskatchewan is 123 students.

Table 1. Province of Saskatchewan 1984 Division IV Enrolments

Location	Schools	% Schools	Student Enrolment	% Students	Average
City Public	24	7	1404	33	585
City Separate	16	5	5169	12	323
City Comp.	7	2	4218	10	603
Rural Sch. Div.	266	76	16784	40	63
Town Boards	9	3	747	2	83
Private/Other	25	7	1604	3	64
Total:					
Non-City	300	86	19135	45	64
City	47	14	23428	55	498
Provincial	347	100	42563	100	123

NOTE: Handbook of Summative Statistics: Division IV Programs. Prepared for The Technical-Vocational/Comprehensive Schools Review. Planning and Evaluation Division: Saskatchewan Education. March, 1986.

With the changes being implemented in the Core Curriculum, Saskatchewan Education estimates that to meet the current university admission requirements, students might conceivably have to take 32 credit units of courses in grades 10 to 12 without taking a single free elective. Considering that the average credit unit offering of smaller schools is about 33 credit units, it is not surprising that these schools offer only the academic program. To illustrate the effect of the university admission standards on a small school, I have included without further comment an actual example of the entire grade 12 course offerings in 1988-89 of a typical low enrolment secondary school (31 grade 12 students) as reported in a computer print-out provided to me by Saskatchewan Education:

<u>Subject</u>	<u>Enrolment</u>
Eng A 30	24
Eng B 30	24
So St 30	28
ECS 30	17
M Alg 30	26
Calculus 30	6
M Geo Trig 30	18
Phys 30	13
Ch 30	20
BSCS 30	25
Sp PE 30	23
Fr 30	10
Acc 36	2
Typ 36	4
IA 35	11

If both Program A and Program B are accepted by the universities as suitable entrance requirements, then the less technical Program A would certainly be the one meeting the needs of the larger number of students and smaller, rural high schools would not be forced to offer the program preparing students for mathematically demanding studies at the post-secondary institutions. Given a choice between the two programs, smaller schools would be able to assign their limited teaching resources to Program A rather than to Program B.

Of course, schools who offer only Program A would have to find the means to offer the more advanced program B to the relatively small minority of mathematically interested and capable students. The successful implementation of both Program A and B will depend upon the leadership of Saskatchewan Education in developing alternative methods of program delivery: That is, creating the means for delivering

the courses in Program B to capable students in smaller secondary schools. Fair and equal educational opportunities must exist for all students regardless of geographical location and, this ideal will not be attained unless serious consideration is given to alternative methods of program delivery for students with different needs, interests and capabilities.

Saskatchewan Education, teachers and administrators must collaboratively develop and experiment with different strategies for program delivery, especially finding suitable means for instructing the minority of students who might desire the challenge of Program B. Alternatives such as the following should be considered: (1) distance education; (2) immersion courses.

Distance education in the form of correspondence courses has been the traditional means of delivering programs to students not having access to qualified teachers. In 1987-88, 10 704 students were enrolled with the Government Correspondence School. Sixty-four per cent of these students are from rural schools.

This distance education technique still has much to offer, especially for those students who can learn with little direct instruction, have a high level of reading comprehension, and have the necessary self-discipline and study habits to persevere without considerable adult supervision and intervention. While correspondence courses have experienced considerable success and growth, the primary delivery system, correspondence through mail, has remained virtually unchanged.

Alberta is one province that has recognized that the development of newer technologies means that distance education must involve more than the creation and delivery of correspondence courses. The Alberta Correspondence School has undertaken a three-year joint venture project with other government agencies and the private sector to design, develop, and implement and market an Advanced Learning System (ALS) for distance learning. Described in the information booklet A Proposal to Develop an Advanced Learning System for the Alberta Correspondence School published in 1988 by the Planning and Policy Secretariat and the Alberta Correspondence School, ALS consists of a number of linked computer databases (curriculum, learning resources, student information, teacher resources, and evaluation) integrated through a sophisticated computer managed learning system.

Currently, in the piloting stage, the developers of ALS intend to produce and manage high-quality student-oriented instructional resources for the following uses:

- a. individual students as in traditional correspondence courses;
- b. low enrolment schools unable to provide the full spectrum of core courses;

c. regular schools interested in alternatives to current text-based program delivery.

In a nutshell, the developers claim that the project will be "technology-linked and would be based on a network based system, involving modularization of instructional programs, high media interaction, built-in self-appraisal mechanisms, challenge/diagnostic assessment, access to teacher and student databases, an intelligent tutorial interactive learning" (p. 1).

Currently, ALS has been developing course materials for the two core mathematics programs Mathematics 10, 20, 30 and Mathematics 13, 23, 33. Each course will comprise of various combinations of management, instruction and media/print materials to meet student needs. The print materials consist of an instruction booklet, practice booklet, assignment booklet, supervisor or teacher booklet, and possibly an enrichment or background (remedial) booklet. A combination of media materials are being developed including audiotapes, videotapes, kits, teleconferencing, and computer diskettes. Each course will be organized into modules that follow a basic pattern: introduction, pretest, lessons, summary and posttest. In addition, course components will include a diagnostic test, challenge test, final test, and appeal test (Alberta Education, 1988).

The aims and objectives of Alberta's distance education project seem to coincide with those of Saskatchewan's. Consequently, Saskatchewan Education should evaluate the merits of ALS and, to avoid any unnecessary duplication, determine the feasibility of using or adapting the course materials to meet the mathematical needs of students in Saskatchewan.

The many variations of immersion programs have proven to be successful for very capable students who can succeed with much less supervision and instruction than the average student and who can learn mathematical ideas rapidly, yet who enjoy and desire the stimulation of other similarly capable and interested students. An immersion program, as the term I've coined suggests, refers to attempts to bring students together for relatively short and intensive periods of instruction. Much of the interest in this method can be attributed to the extensive work of The Study of Mathematically Precocious Youth (SMPY) initiated by Julian Stanley (see Stanley & Benbow, 1983).

In the first stage of SMPY, teachers and administrators in a district are asked to identify 12- to 14-year-old students who learn mathematical ideas rapidly and who can benefit from a program designed to accelerate their learning. Once identified, these capable students are invited to write the SAT-M, Scholastic Achievement Test in Mathematics, a test normally written by 17- and 18-year-olds as an entrance requirement for many American colleges. SMPY uses the SAT-M for several reasons: (1) it has been designed to be relatively curriculum free; (2) it is a timed test containing many difficult

problems; (3) it can be used to predict the likelihood of success in a fast-paced secondary mathematics course.

Students who do well on the test are invited to enrol in a fast-paced mathematics program. Depending upon the number of students and their geographical proximity, these programs have been offered in summers (3- to 6-weeks of daily instruction), after school hours (evening, weekends), and during regular school hours. With some modifications, the SMPY approach could be used to deliver a sound program to a limited number of very able students in Saskatchewan.

At the very least, schools in a geographical district could pool their resources and use the services of an itinerant teacher to deliver some components of Program B. Past experiences with immersion programs indicate that the most able students need less supervision than most students. However, they do need the support of other students and the encouragement of an adult teacher. Consequently, these two elements should be incorporated into the delivery of Program B in smaller schools.

The inability of smaller schools to offer two separate university entrance programs will be the major stumbling block in developing this structure. Until teachers, parents, school boards, and students are convinced that Saskatchewan Education will provide the necessary resources to implement this structure in all schools, the courses in Program B will metamorphose into a vague resemblance of the intended courses and Program A will suffer the fate of the current Alternative Program.

3. Selection of students. Obviously, to ensure the integrity of Program B and the popularity of Program A, entrance into Program B should be carefully controlled and restricted to only those students who have the interest and ability to cope with the challenging mathematical content. Otherwise, as history records, large numbers of students, usually because of pressure from parents, will enrol in what is perceived to be the most prestigious program - in this case, Program B - resulting inevitably in a series of courses that appear in print to be academically sound but in practice are diluted to ensure student success and ultimately satisfying the mathematical needs of few students.

The controlling of student enrolment in Program B is a sensitive area that has a potential for creating controversy. But unless entrance into Program B is confined to the students with greatest mathematical potential, the aims of the program will never be met.

Those educators who object to the principle of screening underlying the dual program structure are reminded that Saskatchewan schools have always streamed students according to ability and interest both by program - General Math, Alternative Math,

Academic Math, Calculus - and by class - Algebra 30C, 30B, 30 Enriched and so forth. Streaming is quite prevalent in the larger urban collegiates.

Moreover, students not enrolled in Program B are not necessarily restricting their career options because, assuming the necessary groundwork with the universities has been completed, they will retain the right to enter university and still have an opportunity to take some form of university mathematics courses. Those rare "late bloomers" who suddenly realize that they want to take more advanced mathematics and science courses and regret their decision to take Program A rather than B have a number of available options. If they have not graduated from secondary school, they can elect to take a transfer course to prepare them for a course in Program B or, if their achievement in Program A is sufficiently high, they might be allowed to enrol in Program B. Those students who graduate from secondary school by completing Program A and are eligible to enter the universities have several choices: (1) to take some courses in Program B offered by the regional colleges; (2) to complete the courses via some form of distance education; (3) to complete preparatory non-credit courses that could be developed by the universities as is the current case in some universities; (4) to write an entrance test in lieu of any make-up courses.

The criteria for selecting students must be carefully regulated to ensure its fairness. A combination of several measures could be used to determine a student's placement: (1) past achievement in middle year mathematics, (2) teacher recommendations, (3) achievement on an entrance exam developed and validated by the province. As students begin to realize that Program B has a different purpose and is considerably more difficult than Program A, a process of self-selection can be expected to occur and the pressure to increase enrollments in Program B should subside.⁹

The proportion of students enrolling and qualifying for Program A and B is obviously dependent on the difficulty of Program B and, consequently, difficult to determine in advance. However, in British Columbia, the most highly academic program attracts about 1/3 of the secondary school students while in Alberta, their newly implemented most highly academic program is hoped to attract only about 1/5 of the student body. At the very least, Program B should be designed for considerably less students than the current 70% of students enrolled in Algebra 30 and the 50% enrolled in Geometry-Trigonometry 30.

4. The number of instructional hours. Both programs should provide students with adequate preparation for post-secondary mathematical studies, yet allow some flexibility for completing other requirements of the Core Curriculum. However, I believe that four to five 100-hour courses could provide sufficient preparation for students who plan to enter mathematically-dependent post-secondary programs.

First, because enrolment in Program B would be restricted to the most mathematically capable and interested students, it would be possible for teachers to present material in greater depth, increase the rate of presentation and to have somewhat higher expectations of student performance without increasing the current levels of instructional time allotted to each course. Second, by excising material not considered as preparation for the post-secondary study of mathematics, less time will be wasted in Program B pursuing "nice but irrelevant" topics. Program B can become more "streamlined" than the current four 1-credit unit courses presently offered. Third, students in Program B will likely complete many of the elective courses to be developed and, consequently, enter the universities having taken more course hours in mathematics or in related subject areas than the majority of students currently do. Fourth, as has been demonstrated in many studies of classroom instruction, the amount of hours of instruction in the implemented curriculum is often far less than in the intended curriculum. For these reasons, I believe that if Program B is properly implemented it will provide students with better preparation in less or the same amount of time as the current program.

Structure 2: The Structures of B.C. and Alberta

British Columbia and Alberta face similar problems in meeting the mathematical needs of secondary school students. In each province a single university entrance program is offered, but "cross-over" courses make transfer from the non-university program possible. Saskatchewan Education could consider adopting similar structures for this province, especially after British Columbia and Alberta have had an adequate opportunity to assess the merits of these structures. For this reason, I have provided an overview of the structure of each province and a brief description of some of the courses.

The British Columbia program. In British Columbia, two sequences of courses have been designed to meet the varying needs of students. The province recognized that the needs of all students could not be met either by a single program for all students or by two separate programs that distinguished between non-university-bound and university-bound students. Figure 1 shows the structure and sequence of the program.

Because Mathematics 11 is considered sufficient preparation for most post-secondary institutions, the concepts and skills considered prerequisite for success in Mathematics 11 can be acquired in two different ways: (1) by completing Mathematics 9 and 10 or (2) by completing Mathematics 9A, 10A, and the transfer course Introductory Mathematics 11.

The prerequisite content for Mathematics 11 is covered more slowly in the Mathematics 9A, 10A and Introductory Mathematics 11 courses than in the Mathematics 9 and 10 courses. Many abstract concepts are delayed until the grade 11 year. The

current guide Mathematics Grades 7-12 published by the Program Development Branch, Ministry of Education (1988b) indicates that the secondary courses are organized around a time allotment of 100 hours each. But according to a Ministry official I contacted, each course is based on the assumption that a minimum of 120 hours is available for instruction.

The content of Mathematics 7, 8, 9, and 10 as well as that in Mathematics 9A and 10A is organized into five strands as illustrated in Table 2.

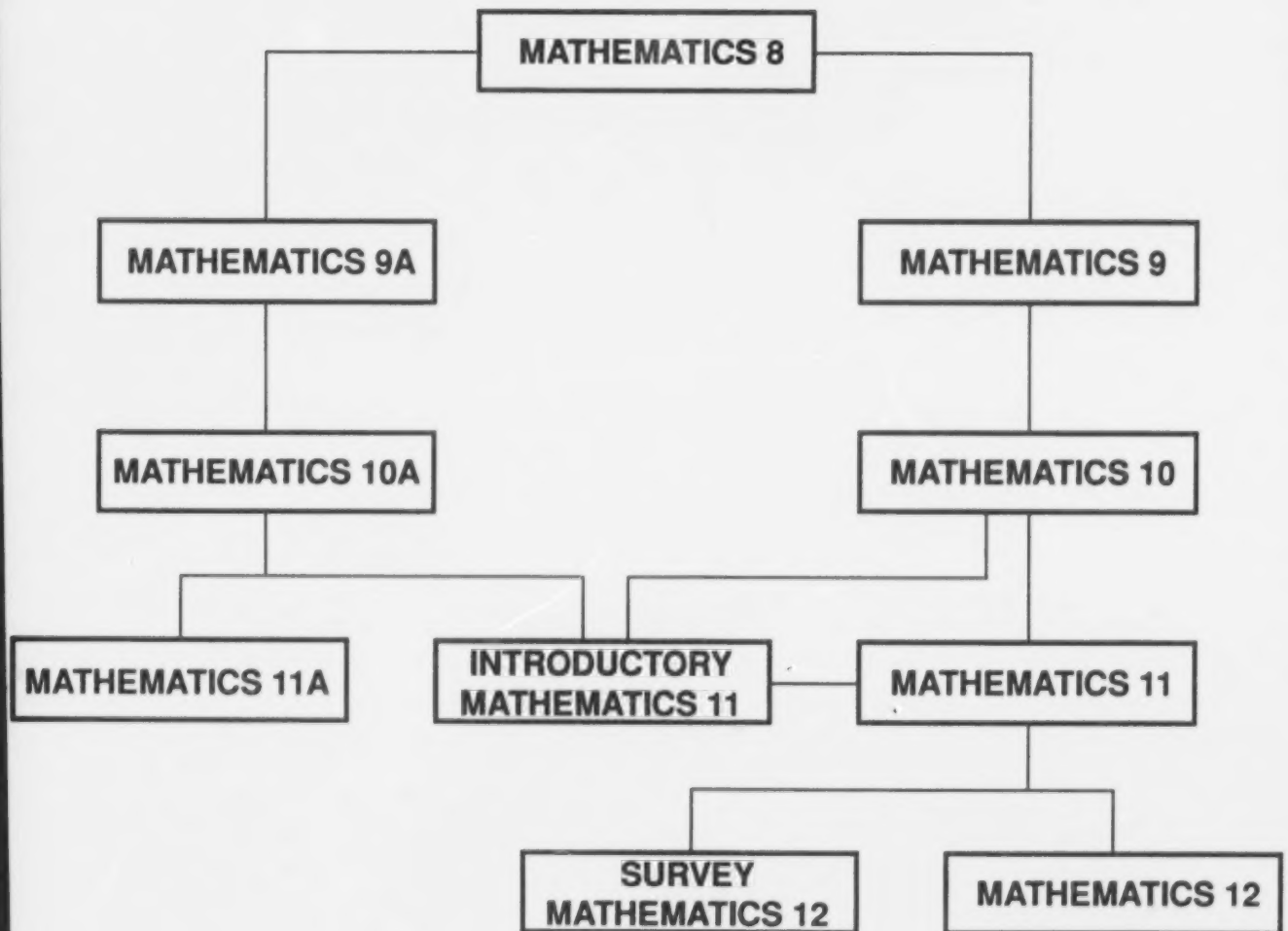


Figure 1. Structure and sequence of the British Columbia secondary mathematics program.

Table 2. Strands and their Estimated Instructional Time of Mathematics Courses in British Columbia

Strand	Estimated number of hours					
	Ma7	Ma8	Ma9	Ma9A	Ma10	Ma10A
Problem-Solving Skills	12	*	*	*	*	*
Number and Number Operations		50	12	40	10	30
Whole numbers	15					
Rational numbers	20					
Integers	10					
Data Analysis	10	10	15	15	10	20
Geometry	12	20	28	15	40	20
Measurement	13	8	10	15	—	15
Algebra	8	12	35	15	40	15

* Intended to be incorporated into all strands

The proposed implementation schedule is as follows: grade 9, 10 and 11, September, 1989; grade 12, September 1990. The Ministry official confirmed that the implementation is on schedule and students in grades 9 and 10 began the new program in September, 1989. Although he did not have access to the annual statistics, he estimated that about two thirds of the students are enrolling in the Math 9 and 10 courses.

Mathematics 11, the university entrance course, is identified as algebra, relations and functions, analytic geometry, trigonometry, measurement, and data analysis. Mathematics 12 is designed for students who intend to study sciences and engineering at the university and is a prerequisite for the introductory calculus course. The following content has been proposed: Trigonometry, conics, complex numbers, equations and inequalities, calculus, problem solving and applications. The Survey Math 12 is a new course intended for students bound for post-secondary studies in mathematics and the humanities and will include content from finite math, probability and statistics, and computer applications. Students with a strong interest in mathematics will be encouraged to take both Mathematics 12 and Survey Math 12.

The Ministry official indicated that Survey Math 12 is meeting with resistance from some university professors who don't believe discrete mathematics is a suitable high school course or who believe that the course may duplicate the content of some

university courses. He also mentioned that smaller, rural schools have complained about the dual program structure and the Ministry is aware that there is a need to improve program delivery.

He also explained that a Departmental Exam prepared by a committee of teachers is administered annually to grade 12 students. He believes strongly that this evaluation system is more of an influence on what is actually taught by teachers than the curriculum guide. He also indicated that the costs for this provincial evaluation process are much higher than the budget provided for curriculum and instruction.

2. The Alberta program. Alberta has developed a rather complex structure to ensure that students can transfer from one program to another.¹⁰ The program structure and recommended transfer points are illustrated in Figure 2.

The Mathematics 10-20-30 and 13-23-33 course sequences originally developed in 1983 provide differentiated instruction to accommodate for varying mathematical abilities and the background required for specific career patterns. Each program route varies in the mathematics subject matter presented, the instructional emphasis, and the time spent on concept development. However, certain topics and objectives are common to both course sequences.

Completion of Mathematics 20 fulfills the mathematics requirement to enter a university in Alberta: however, few options are left open for such students. Mathematics 31 is intended for students who require more advanced preparation for entrance into selected post-secondary programs such as engineering. The current content of Mathematics 31 is listed below:

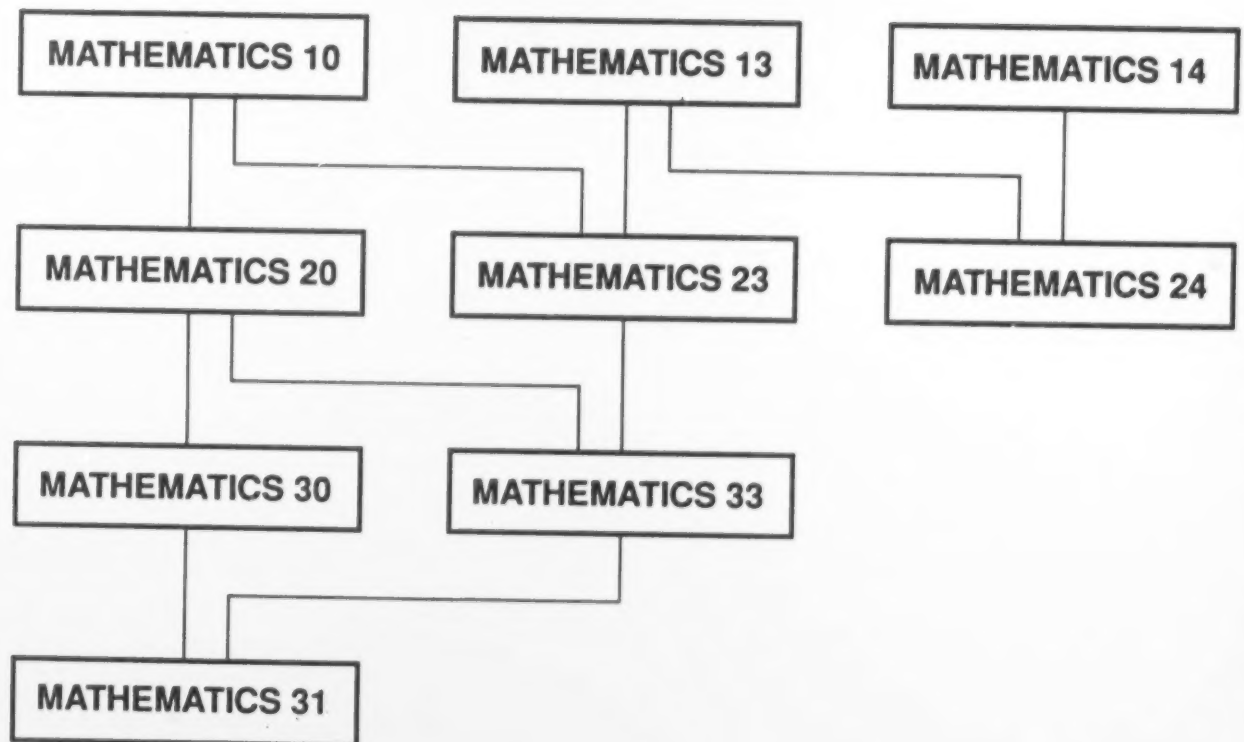


Figure 2. Structure and sequence of the Alberta secondary mathematics program.

Calculus (50%): Derivatives of algebraic relations; applications of derivatives; integration.

Vectors (30%): Geometric vectors and their applications; algebraic vectors and their applications; dot product.

Electives (20%): Volumes of revolution; calculus of logarithmic functions; trigonometric functions; systems of linear equations; matrices and linear transformations; polar coordinates.

Mathematics 14 and 24 are general math courses concentrating on developing fundamental mathematics skills and their application in consumer, technical and service contexts.

All courses are currently under review. However, only minor modifications to the program structure are being considered. As of February 1988, draft programs of studies for Mathematics 10-20-30 and 13-23-33 have been distributed to all high schools and post-secondary institutions for review. Each grade level course will be implemented year by year until the entire implementation is completed in 1991. Textbooks are currently being piloted.

Judging from the student enrolment in the secondary mathematics programs in Alberta shown in Table 3, both programs attract a considerable number of students. Such a balance in student numbers contrasts sharply with the imbalance in Saskatchewan. However, a Department official admitted that Alberta has not been able to obtain the desired "20-60-20" enrolment distribution (20% enrolled in Math 10, 60% enrolled in Math 13 and 20% enrolled in Math 14). He believes the current university entrance standards and credentialing are the major reasons for the over-enrolment in the 10-20-30 series of courses. In examining the content of both programs, I surmise that students may have discovered that the content differences between the 13-23-33 and 10-20-30 programs are not that substantial and, consequently, choose the 10-20-30 program to increase their future options. This same official stated that rural schools usually offer only the 10-20-30 program and assign lower achievers in these course a grade in the 13-23-33 courses. He confirmed that Alberta Education has budgeted a great deal of money to provide "equity programs" to help lower enrolment schools offer a variety of programs.

Table 3. Student Enrolment in the Alberta Secondary Mathematic Courses 1986-87

Subject	No. of schools	No. of students	Percent of students
Mathematics 10	345	21 141	53
Mathematics 13	270	12 501	31
Mathematics 15	257	6 489	16
Mathematics 20	321	18 650	58
Mathematics 23	254	11 163	35
Mathematics 25	172	2 209	7
Mathematics 30	319	19 677	54
Mathematics 31	210	5 023	14
Mathematics 33	226	8 997	27

Source: 82nd Annual Report, 1986-87, Alberta Education

Structure 3: A Common Multiple Exit program

Perhaps the most practical structure to be recommended is a single program with multiple exits as illustrated in Figure 3. Each 1-credit course would consist of 100 instructional hours as represented by the designators Mathematics 100, 200, and so forth. The program would provide students with a

number of exits: Math 200 to graduate from high school; Math 300 to meet the requirements for general university entrance; Math 400 and likely 500 to meet the requirements for taking the introductory post-secondary mathematics courses and enter mathematically-dependent post-secondary programs.

The content of each course would be determined in several stages: (1) by identifying the **basic** knowledge, skills, and understandings needed to function in an introductory post-secondary mathematics course or in a mathematically-dependent field; (2) by identifying the skills and understandings needed by a numerate person.

The pace of presentation and the amount of post-secondary preparatory work would gradually increase in each successive course. Consequently, the less able and disinterested students would drop out earlier in the program and the enrolment in each

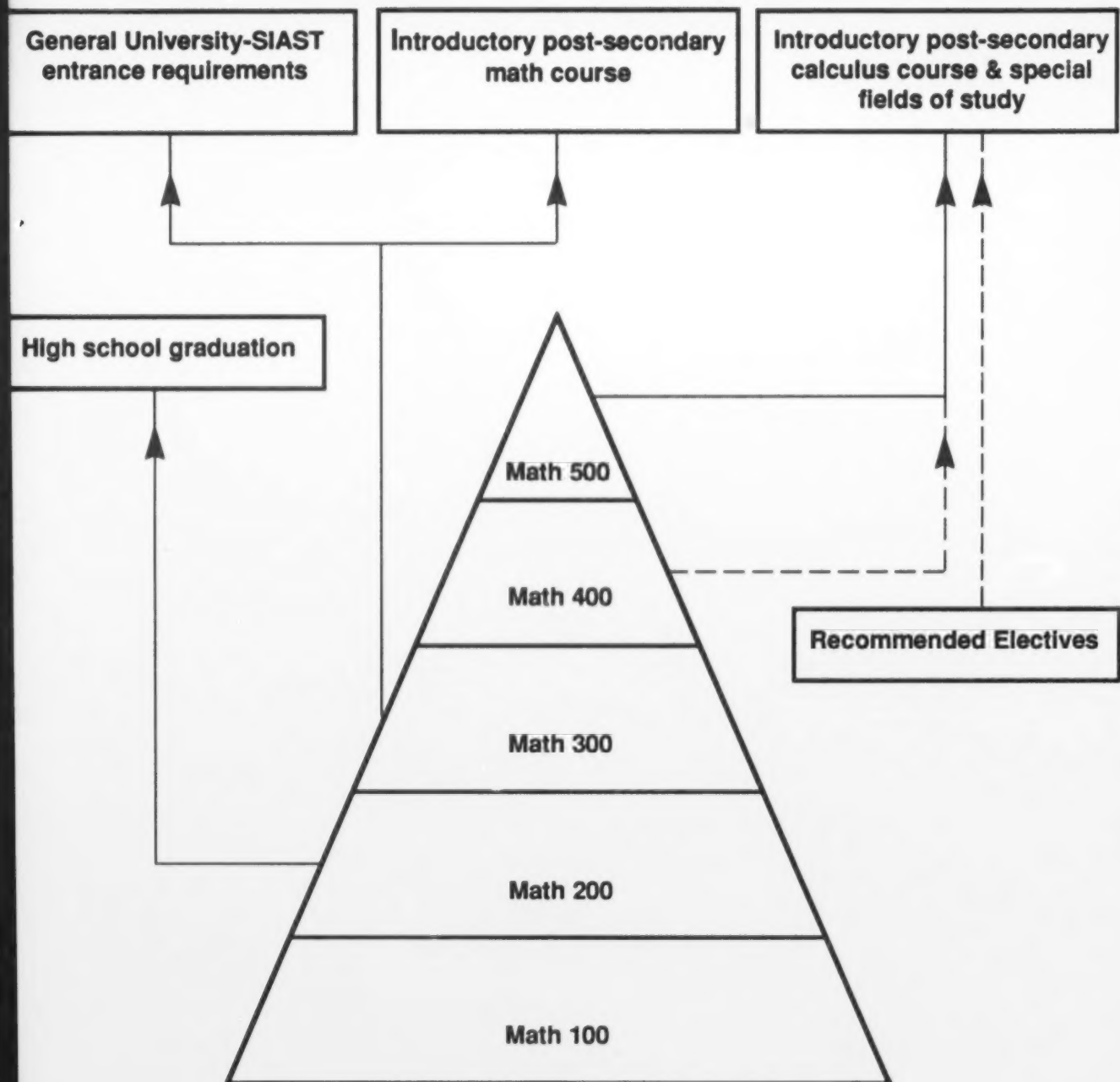


Figure 3. A single program structure with multiple exits.

course would become successively smaller. These students would be given the opportunity to take additional more practically oriented elective courses offered by Saskatchewan Education or by the local district.

The triangular shape of the diagram is intended to represent both the student enrolment and the amount of post-secondary preparatory work in each successive course. The sections closer to the base of the triangle represent courses that are intended to be broad in nature and contain a variety of topics: advanced arithmetic (estimation, mental calculation, percentage, rational numbers, ratio, and proportion), measurement, data organization and analysis, probability, statistics, geometry, an introduction to trigonometry and logarithms, and algebra. In these introductory courses, the emphasis would be more on the development of numeracy than on the acquisition of a number of mathematical techniques. Students would learn how these mathematical ideas apply to the real as well as the mathematical world.

The sections in the diagram approaching the apex represent progressively more specialized courses. The intent of the last two courses is to prepare students for the introductory mathematical work at the universities and other post-secondary institutions. In this sense, the content included in these later courses would be less "practical" than in the earlier courses.

Each course could be partitioned into modules to ensure that students receive some instruction in mathematics in each year of high school. Otherwise some students in a semestered school could theoretically complete the mathematical requirements (Math 100 and 200) by the end of grade 10, which I should emphasise is possible to do in the present program, and not take any further courses in their last two years of school.

Each course is assumed to be a prerequisite for the next course in the sequence. This hierarchical progression will be a problem for students in unsemestered schools who intend to complete Math 400 or 500. Either all schools will have to be semestered or the structure will have to be modified to allow students to take the upper level courses concurrently.¹¹

As with the dual program structure discussed previously, the entrance requirements of the universities would have to be reconsidered. Saskatchewan Education would have to collaborate with the post-secondary institutions to determine the content expectations for students entering various programs in order to determine the most appropriate content for each course, especially Mathematics 400 and 500. Elective courses similar to those recommended for the dual program structure discussed previously would be developed by Saskatchewan Education to provide further study for mathematically able and interested students. Some post-secondary programs would likely strongly recommend but not require these electives as an entrance requirement.

The nature of the electives should comply with the position statement of the National Council of Teachers of Mathematics regarding provisions for mathematically talented and gifted students:

The needs of the mathematically talented and gifted students cannot be met by programs of study that only accelerate these students through the standard school curriculum, nor can they be met by programs that allow students to terminate their study of mathematics before their graduation from high school. The curriculum should provide for all mathematically talented and gifted students every year they are in school. These students need enriched and expanded curricula that emphasize higher order thinking skills, non-traditional topics, and applications of skills and concepts in a variety of contexts.

Therefore, the National Council of Teachers of Mathematics recommends that all mathematically talented and gifted students should be enrolled in a program that provides a broad and enriched view of mathematics in a context of higher expectations. Acceleration within such a program is recommended only for those students whose interests, attitudes, and participation clearly reflect the ability to persevere and excel throughout the entire program. (NCTM, 1987, p. 100)

This single program structure has several advantages over other structures. First, it is suited to the lower enrolment secondary schools. They could assign teachers to teach Mathematics 100 to 400, as they currently do with the Academic Program. Arrangements would have to be made to ensure that a minority of students have access to Math 500 and any electives that might be developed.

I recommend that Saskatchewan Education work closely with small rural schools to develop creative ways to deliver the upper level courses and the electives. It is also essential that Saskatchewan Education identify and evaluate the current approaches used by small schools to deliver senior level mathematics courses. I recommend that the secondary writer conduct either a review of recent literature or a thorough survey to gather this information on the course delivery practices of schools.¹²

Second, as the courses increase in mathematical difficulty, only the most capable and most interested students would continue on in the program and complete the last two courses Mathematics 400 and 500. Thus, there would be no need to select students *a priori*. Instead, student progress in the sequence would be determined on the basis of success in a previous course rather than on some other more questionable placement criteria. Third, a single program structure with multiple exits reduces the stigmatization normally associated with "tracking." Students would exit from the program at different levels but they would be enrolled in the same program.

The success of this program would depend largely on the number of students taking Mathematics 400 and 500. If the enrolment in Mathematics 400 and 500 approaches the current levels in Algebra 30 and Geometry-Trigonometry 30 (about 70% and 50% of grade 12 students, respectively), then the courses will not be able to adequately prepare students for post-secondary studies in mathematics. I assume that only about 20% to 25% of grade 12 students would be enrolled in Mathematics 400. Perhaps the enrolment in these mathematics courses should be no higher than in other senior-level, academic school subjects, French 30, for example, whose enrolment in 1987-88 was about 2400 or less than 20% of grade 12 students.

Ensuring that Mathematics 400 and 500 contain difficult mathematical ideas is the only practical way to limit student enrolment. The enrolment in these courses will become reasonably low once students and counsellors realize that only the most capable and most interested students are likely to succeed. Of course, if the universities do not change their entrance requirements, then enrolment in Mathematics 400 and 500 will be about the same as present.

The B.C. Royal Commission has recommended that a structure similar to my proposed structure be used to organize all school subjects. In grades 1 to 10 a Common Curriculum will be developed consisting of four categories of subject matter: Humanities, Fine Arts, Sciences, and Practical Arts (Mathematics is included under the category of Sciences). In grades 11 and 12, students would take a core program consisting of courses in English (language and literature) and History (Canadian and World), and a course in Science, Technology and the Environment. They also recommended that Physical Education be considered a core component.

Students will also be expected to take those grade 11 and 12 subjects required for specific post-secondary or career programs. As discussed previously in this chapter, the report emphasises that a major obstacle to the successful implementation of their proposals is the current university entrance requirements. They suggest that the Ministry of Education collaborate with the universities to determine the content of the grade 11 and 12 academic courses considered as entrance requirements.

In summary, the structure, purposes, nature, and content of the secondary mathematics program need to be changed significantly. However, the necessary changes will not occur unless Saskatchewan Education proceeds in a gradual and cautious manner and collaborates from the beginning with secondary mathematics teachers, small rural high schools, the post-secondary institutions, especially the universities, and other stakeholders in the development of a new mathematics program. It is very unlikely that the needed changes can be completed within the time period currently proposed by Saskatchewan Education for development, piloting and implementation. Consequently, I recommend that the current schedule be lengthened to ensure that substantial rather than cosmetic changes can be made.

Footnotes

- ¹ An incredible amount of knowledge of mathematics, science and technology is needed to design a microwave, a good knowledge is needed to understand its operation, and almost no knowledge is needed to operate it.
- ² The terms Alternative and Academic need to be replaced with more appropriate descriptors. The majority of the topics found in the present Alternative Program can be labelled "academic." For the purposes of this report, I will use the descriptors "A" and "B." I have also assumed that all courses will be assigned 1 credit unit and 100 instructional hours to comply with Saskatchewan Education policy.
- ³ Saskatchewan is the only remaining province that offers separate courses in Algebra and Geometry-Trigonometry. All provinces except for Ontario offer integrated mathematics courses, while Ontario offers senior courses in Discrete Math and Calculus, and a combined course of Algebra and Geometry. If Saskatchewan continues to offer separate courses in Algebra and Geometry-Trigonometry, American rather than Canadian texts will have to be authorized. However, if the NCTM Standards has an influence on curriculum development in the U.S.A, even this source will disappear.
- ⁴ Stanley and Benbow (1982) estimate from research done with mathematically precocious youth (SMPY) that about 36 000 11-, 12- and 13-year-olds in the U.S.A. reason so well mathematically that they score higher on a standard-ized algebra test than students who have studied algebra for a full school year. Thus, on a proportionate basis, there are likely several hundred middle grade students in any given year in Saskatchewan who are similarly capable yet destined to study mathematics at a pace and level unsuited to their abilities.
- ⁵ Recently, a principal of a large Collegiate told me that he was bothered by the tendency of his mathematics department not to assign grades higher than 78% to students in Algebra 30 (Enriched). He commented that the average grades for students in other Algebra 30 classes are slightly higher than students in the Enriched class.
- ⁶ I witnessed what I believe is the arbitrariness of some actions taken by University Council, the academic decision-making body of the University of Saskatchewan. In considering a proposal for a new Nursing program, Council debated the wisdom of requiring Physics 30 as an entrance requirement for the College of Nursing. Despite the opposition of the Nursing faculty, Council added this entrance requirement. The meeting was poorly attended and less than 50 faculty were able to impose a requirement that obviously will have considerable impact on secondary students planning to enrol in the College of Nursing. The effect of this requirement on secondary schools, especially on smaller rural schools, was never raised during Council's debate.
- ⁷ The College of Education insists that all prospective elementary school teachers complete Mathematics 102.6. The faculty in the College who support this requirement typically have little or no knowledge of the content. Usually when I ask them to justify this requirement, they say something vague like "All teachers should have a university course

in mathematics." Obviously this form of reasoning could be used to justify almost any course listed in the University Calendar.

- ⁸ Saskatchewan Education currently lists 43 half credit courses with a student enrolment of approximately 12 000 students.
- ⁹ Consider what happened to the enrolment in courses such as Latin when students were given other options to satisfy university entrance requirements.
- ¹⁰ Alberta currently uses the following last digit codes to specify different programs: 0 = most academically challenging; 1 = differentiated in terms of curriculum expectations; 2 = vocational courses; 3 = less academic challenging than 0 courses; 4 = less academic challenging than 3 courses; 5 = locally developed courses; 6 = Integrated Occupational Program Courses.
- ¹¹ In 1989-90, 192 and 119 grade 9-12 schools were semestered and unsemestered respectively. The 100 instructional hours per 1 credit unit mandated by Saskatchewan Education is more suited to semestered than unsemestered schools.
- ¹² Some small schools are using a "multiple class" organization to deliver several senior mathematics courses: in other words, assigning two or more courses to the same instructional period. For example, one member of the Advisory Committee is currently teaching Algebra 30 and General Math 30 in the same class period to two different groups of students. I've been told that some schools increase their course offerings by teaching an academic course such as Algebra 30 to all grade 12 students and giving credit for Mathematics 30 to those students who do not perform well.

REFERENCES

- Advisory Council for Adult and Continuing [The ACACE study] (1982). *Adults' mathematical ability and performanc*. Leicester, England: ACACE
- Association for Supervision and Curriculum Development [ASCD] (1988). *Dimensions of thinking. A framework for curriculum and instruction*. Alexandria, VA: ASCD.
- Baroody, A.J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers*. New York: Teachers College Press, Columbia University.
- Baroody, A.J., Ginsburg, H.P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education*, 14, 156-168.
- Beamer, J. E. (1976). *Elementary school mathematics assessment project, Volume 1*. Saskatoon, SASK: University of Saskatchewan.
- Bell, A., Swan, M., & Taylor, G. (1981). The choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics*, 12, 399-420.
- Benjamin, H. (1939). *The sabre tooth curriculum*. New York: McGraw-Hill Book Company.
- Bernard, J.E., & Cohen, M.P. (1988). An integration of equation-solving methods into a developmental learning sequence. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 97-111). Reston, VA: National Council of Teachers of Mathematics.
- Bishop, A.J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 175-203). New York: Academic Press.
- Booth, L.R. (1988). Children's difficulties in beginning algebra. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 20-32). Reston, VA: National Council of Teachers of Mathematics.
- Borasi, R. (1986, May). On the nature of problems. *Educational Studies in Mathematics*, 17(2), 125-141.

- Borenson, H. (1985). *The hands-on equations learning system: Introduction and level I*. Penndel, PA: Borenson and Associates.
- British Columbia Ministry of Education (1988a). *Mathematics 1-8 Curriculum Guide*. Victoria, B.C.: Ministry of Education.
- British Columbia Ministry of Education (1988b). *Mathematics: Grade 7-12*. Victoria, B.C.: Ministry of Education.
- British Columbia Royal Commission of Education (1988). *A legacy for learners*. Victoria, B.C.: Ministry of Education.
- Burger, W.F., & Shaughnessy, J.M. (1986, January). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.
- Cambridge Conference (1963). *Goals for school mathematics*. New York, NY: Houghton Mifflin.
- Capon, N., & Kuhn, D. (1979). Logical reasoning in the supermarket: Adult females' use of a proportional reasoning strategy in an everyday context. *Developmental Psychology*, 15(4), 450-452.
- Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985a). Written and oral mathematics. *Journal for Research in Mathematics Education*, 18(2), 83-97.
- Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985b). Mathematics in the streets and in the schools. *British Journal of Developmental Psychology*, 3, 21-29.
- Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Eds.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17-40). Hillsdale, N.J.: Lawrence Erlbaum Associates.
- Carpenter, T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M., & Reys, R.E. (1981). *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.
- Carpenter, T. P., Hiebert, J., & Moser, J. M. (1983). The effects of instruction on children's solutions of addition and subtraction word problems. *Educational Studies in Mathematics*, 14, 55-72.

- Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15, 179-202.
- Chaffe-Stengel, P., & Noddings, N. (1982). Facilitating symbolic understanding of fractions. *For the Learning of Mathematics*, 3, 42-48.
- Chalouh, L., & Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 33-42). Reston, VA: National Council of Teachers of Mathematics.
- Charles, R.I., & Lester, F.K. (1982). *Teaching problem solving: What, why & how*. Palo Alto, CA: Dale Seymour Publications.
- Charles, R.I., & Lester, F.K. (1984). An evaluation of a process-oriented mathematical problem-solving instructional program in grades five and seven. *Journal for Research in Mathematics Education*, 15, 15-34.
- Cipywnyk, S.V., Pawlovich, W.E., & Randhawa, B.S. (1983, March). *Early school leavers in Saskatchewan: A preliminary study*. Saskatoon, SASK: University of Saskatchewan.
- Clay, G.V., & Kolb, J.R. (1983). The acquisition of meanings for arithmetic symbols. *Proceedings of the Fifth Annual Meeting, Volume 1*. PME-NA, Montreal, Quebec.
- Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics*, 16, 111-126.
- Cobb, P. (1986, June). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics*, 6(2), 2-9.
- Committee of Inquiry into the Teaching of Mathematics in Schools [Cockcroft Report] (1983). *Mathematics counts*. London: HMSO.
- Crowley, M. L. (1987). The van Hiele model of development. In M. M. Lindquist (Ed.), *Learning and teaching geometry, K - 12* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.
- Davis, R.B. (1989). Three ways of improving cognitive studies in algebra. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 115-119). Reston, VA: National Council of Teachers of Mathematics.
- Hono, E. (1979). *Future positive*. London: Maurice Temple Smith.

- De la Rocha, O. (1985, Fall). The reorganization of arithmetic practice in the kitchen. *Anthropology and Education Quarterly*, 16(3), 193-198.
- Del Grande, J.J. (1987). Spatial perception and primary geometry. In M. M. Lindquist (Ed.), *Learning and teaching geometry, K - 12* (pp. 126-135). Reston, VA: National Council of Teachers of Mathematics.
- Desforges, C., & Cockburn, A. (1987). *Understanding the mathematics teacher: A study of practice in first schools*. London: The Falmer Press.
- Ekenstam, A.A., & Nilsson, M. (1979). A new approach to the assessment of children's mathematical competence. *Educational Studies in Mathematics*, 10, 41-66.
- Fennema, E., & Tartre, L.A. (1985, May). The use of spatial visualization in mathematics by girls and boys. *Journal for Research in Mathematics Education*, 16(3), 184-206.
- Fey, J.T. (1981). *Mathematics teaching today: Perspectives from three national surveys*. Reston, VA: National Council of Teachers of Mathematics.
- Fey, J. (1989). School algebra for the year 2000. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 199-213). Reston, VA: National Council of Teachers of Mathematics.
- Fischbein, E., Deri, M., Nello, M.S., & Marion, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16(1), 3-17.
- Flanders, J.R. (1987). How much of the content in mathematics is new? *Arithmetic Teacher*, 31(1), 18-23.
- Fuson, K.C., & Brinko, K.T. (1985, May). The comparative effectiveness of microcomputers and flash cards in the drill and practice of basic mathematics facts. *Journal for Research in Mathematics Education*, 16(3), 225-232.
- Giesbrecht, E. C. (1977). *The attainment of selected mathematical competencies by high school students in Saskatchewan*. Unpublished doctoral dissertation, University of Saskatchewan.
- Hamann, M.S., & Ashcraft, M.H. (1986). Textbook presentations of the basic addition facts. *Cognition and Instruction*, 3, 173-192.
- Hardy, G.A. (1940). *A mathematician's apology*. Cambridge, England: The University Press.

- Hart, K. (1982). Mistakes in mathematics. The research of CSMS. In I.D. Beattie, T. Bates, J. Sherrill, & D. Owens (Eds.), *Selected proceedings of research reports: 1981 research monograph* (pp. 111-117). Vancouver, B.C.: Research Council for Diagnostic and Prescriptive Mathematics.
- Hembree, R., & Dessart, D.J. (1986). Effects of hand-held calculators in precollege mathematics education: A meta-analysis. *Journal for Research in Mathematics Education*, 17, 83-89.
- Hendrickson, A. D. (1986, April). Verbal multiplication and division problems: Some difficulties and some solutions. *Arithmetic Teacher*, 33, 26-33.
- Herscovics, N. (1989). Cognitive obstacles in the learning of algebra. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics.
- Hiebert, J. (1984). Children's mathematics learning: The struggle to link form and understanding. *The Elementary School Journal*, 84(5), 407-513.
- Hill, S. (1982). The case for a new high school mathematics curriculum. In S. Hill (Ed.), *Education in the 80's: Mathematics* (pp. 108-117). Washington, D.C.: National Education Association.
- Hiscox, M.A.B. (1980). *The mathematical requirements of the pre-'A' level leaver*. Nottingham, England: The Shell Centre for Mathematical Education, The University of Nottingham.
- Hoffer, A. (1983). Van Hiele-based research.. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205-227). New York: Academic Press.
- Hoffer, A.R. (1988). Ratios and proportional thinking. In T.R. Post (Ed.), *Teaching mathematics in grades K-8. Research based methods* (pp. 285-313). Toronto, ONT: Allyn & Bacon, Inc..
- Hope, J.A. (1986). Mental calculation: Anachronism or basic skill? In H.L. Schoen & M.J. Zweng (Eds.), *Estimation and mental computation* (pp. 45-54). Reston, VA: National Council of Teachers of Mathematics.
- Hope, J.A. (1987a). A case study of a highly skilled mental calculator. *Journal for Research in Mathematics Education*, 18(5), 331-342.
- Hope, J.A. (1987b). *Numeracy: A common essential learning for Saskatchewan students*. Regina, Sask: Saskatchewan Education.

- Hope, J.A., & Baker, J. (1986). *Focus on problem solving*. Saskatoon, SK: Saskatoon Board of Education.
- Hope, J. A., Baker, J., & Mansfield, E. (1987) A study of the ability to recall basic number facts. In *Academic goals study: An exploration of the attainment of communication, computation, critical and creative thinking, and lifelong learning skills*, (pp. 29-46). Saskatoon, SASK: Educational Services Department, Saskatoon Board of Education.
- Hope, J.A., & Kelly, I.W. (1983, November). Common difficulties with probabilistic reasoning. *Mathematics Teacher*, 76(8), 565-570.
- Hope, J.A., & Owens, D.T. (1987). An analysis of the difficulty of learning fractions. *Focus on Learning Problems in Mathematics*, 9(4), 25-40.
- Hope, J.A., Reys, R.E., & Reys, B. (1990). Mental computation - A snapshot of second and fifth grade student performance. Unpublished study.
- Hope, J.A., & Sherrill, J.E. (1987). Characteristics of unskilled and skilled mental calculators. *Journal for Research in Mathematics Education*, 18(2), 98-111.
- Huff, D. (1954). *How to lie with statistics*. New York: W.W. Norton & Co.
- Hughes, M. (1988). *Children and number: Difficulties in learning mathematics*. Oxford, U.K.: Basil Blackwell Ltd.
- Inskeep, J.E. (1976). Teaching measurement to elementary school children. In D. Nelson & R.E. Reys (Eds.), *Measurement in school mathematics* (pp. 60- 86). Reston, VA: National Council of Teachers of Mathematics.
- Jones, P.S., & Coxford, A.F. (1970). From Colburn to the rise of the universities: 1821-94. In P.S. Jones (Ed.), *A history of mathematics education in the United States and Canada* (pp. 24-35). Reston, VA: National Council of Teachers of Mathematics.
- Kahneman, D., & Tversky, A. (1982). On the study of statistical intuitions. In D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgement under certainty: Heuristics and biases*. Cambridge: Cambridge University Press.
- Kamii, C. (1985, September). Leading primary education toward excellence: Beyond worksheets and drill. *Young Children*, 40(6), 3-9.
- Kapoor, D. (1974, Fall). Alternate Division IV (Grades 10 to 12). *Saskatchewan Mathematics Teachers Society News/Journal*, 13(1), 10-13, 16.

- Karplus, E., Karplus, R., & Wollman, W. (1974, October). Intellectual development beyond elementary school: Ratio, the influence of cognitive style (Vol 4). *School Science and Mathematics*, 74, 475-482.
- Karplus, R., Pulos, S., & Stage, Elizabeth, E.K. (1983). In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45-90). New York: Academic Press.
- Kastner, B. (1989, February). Number sense: The role of measurement applications. *Arithmetic Teacher*, 36(6), 40-46.
- Keitel, C. (1988). Mathematics education and technology. In L. Pereira-Mendoza (Ed.), *Proceedings 1988 Annual Meeting Canadian Mathematics Education Study Group* (pp. 1-15). Winnipeg: CMESG
- Kieren, C. (1988). Two different approaches among algebra learners. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 91-96). Reston, VA: National Council of Teachers of Mathematics.
- Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33-56). Reston, VA: National Council of Teachers of Mathematics.
- Kieren, T.E. & Olson, A.T. (1989, January). Imagination, intuition, and computing in school algebra. *Arithmetic Teacher*, 82(1), 14-17.
- Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1-16). Hillsdale, NJ: Lawrence Erlbaum Assoc.
- Kouba, V.L., Brown, C.A., Carpenter, T.P., Lindquist, M.M., Silver, E.A., & Swafford, J.O. (1988, May). Results of the Fourth NAEP Assessment of Mathematics: Measurement, geometry, data interpretation, attitudes, and other topics. *Arithmetic Teacher*, 35(9), 10-16.
- Krulik, S., & Rudnick, J.A. (1980). *Problem solving: A handbook for teachers*. Toronto, ONT: Allyn & Bacon, Inc.
- Krutetskii, V.A. (1976). *The psychology of mathematical abilities in schoolchildren*. Edited by J. Kilpatrick and I. Wirszup. Chicago, IL: University of Chicago Press.

- Lappan, G., & Schram, P.W. (1989). Communication and reasoning: Critical dimensions of sense making in mathematics. In P.R. Trafton and A.P. Shulte (Eds.), *New directions for elementary school mathematics* (pp. 14-30). Reston, VA: National Council of Teachers of Mathematics.
- Leitzel, J.R. (1989). Critical considerations for the future of algebra instruction. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 25-32). Reston, VA: National Council of Teachers of Mathematics.
- Lester, F. K. Jr. (1980). Problem solving: Is it a problem? In M. M. Lindquist (Ed.), *Selected issues in mathematics education* (pp. 29-45). Chicago, ILL: National Society for the Study of Education.
- Lester, F. K. (1983). Trends and issues in mathematical problem solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 229-261). New York: Academic Press.
- Lester, F.K. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41-69). Hillsdale, NJ: Lawrence Erlbaum Assoc.
- Levinson, P. (1986). Information technologies as vehicles of evolution. In Carl Mitcham & Alois Huning (Eds.), *Philosophy and Technology II* (pp. 29-47). Boston, MA: D. Reidel Publishing Co.
- Lochhead, J., & Mestre, J.P. (1988). From words to algebra: Mending misconceptions. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 127-135). Reston, VA: National Council of Teachers of Mathematics.
- Lovell, K. (1971). *The growth of understanding in mathematics: Kindergarten through grade three*. New York: Holt, Rinehart & Winston.
- Markovits, Z. (1989). Reactions to the number sense conference. In J. Sowder (Ed.), *Establishing foundations for research on number sense and related topics: Report of a conference*. San Diego, CA: Center for Research in Mathematics and Science Education.
- Markovits, Z., Eylon, B.S., & Bruckheimer, M. (1988). Difficulties students have with the function concept. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 43-60). Reston, VA: National Council of Teachers of Mathematics.

- Mathematical Sciences Education Board (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, D.C.: National Academy Press.
- McIntosh, A. T. (1976). *Mathematical requirements of employers of lower ability school leavers*. Leicestershire, England: Leicestershire Education Department.
- Morris, A., & Cope, R.G. (1982). *Literacy and numeracy demands on school leavers entering selected apprenticeships in Queensland, Australia. Volume I and II*. Brisbane, Australia: Brisbane College of Advanced Education. (Eric microfiche ED224975)
- Murtaugh, M. (1985, Fall). The practice of arithmetic by American grocery shoppers. *Anthropology and Education Quarterly*, 16(3), 186-192.
- National Advisory Committee on Mathematical Education [NACOME] (1975). *Overview and analysis of school mathematics: Grades K-12*. Washington, D.C.: Conference Board of the Mathematical Sciences.
- National Assessment of Educational Progress [NAEP] (1977). *Math technical report: Exercise volume*. Denver, CO: Educational Commission of the States.
- National Assessment of Educational Progress [NAEP] (1979). *Mathematical understanding*. Denver, CO: Educational Commission of the States.
- National Assessment of Educational Progress [NAEP] (1982). *The Third National Mathematics Assessment: Results, trends, and issues*. Denver, CO: Educational Commission of the States.
- National Council of Supervisors of Mathematics (1989, May). Essential mathematics for the twenty-first century. The position of the National Council of Supervisors of Mathematics. *Arithmetic Teacher*, 36(9), 27-29.
- National Council of Teachers of Mathematics (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (1987, October). *Curriculum and evaluation standards for school mathematics. A working draft*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (1989, March). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

- National Science Foundation - Department of Education (1980). *Science and engineering education for the 1980's and beyond*. Washington, D.C.: U.S. Government Printing Office.
- O'Brien, T.C., & Casey, S.A. (1983, March). Children learning multiplication, Part 1. *School Science and Mathematics*, 83, 246-251.
- O'Daffer, P.G. (1980). Geometry: What shape for a comprehensive, balanced curriculum? In M.M. Lindquist (Ed.), *Selected issues in mathematics education* (pp. 90-105). Berkeley, CA: McCutchan Publishing.
- Parker, T. (1984). *In one day*. New York, NY: Houghton Mifflin.
- Paulos, J.A. (1988). *Innumeracy: Mathematical illiteracy and its consequences*. New York, NY: Hill and Wang.
- Payne, J.N. (1984, February). Curricular issues: Teaching rational numbers. *Arithmetic Teacher*, 31, 14-19.
- Peck, D.M., & Jencks, S.M. (1981). Conceptual issues in the teaching and learning of fractions. *Journal for Research in Mathematics Education*, 12(5), 339-348.
- Pirie, S. (1981). *Mathematics in medicine: A report for the Cockcroft committee*. Nottingham, England: The Shell Centre for Mathematical Education, The University of Nottingham.
- Randhawa, B.S. (1986). *Academic basic skills of grades four, seven, and ten students of Saskatchewan in 1985*. Saskatoon, SASK: University of Saskatchewan.
- Resnick, L.B. (1986). The development of mathematical intuition. In M. Perimutter (Ed.), *Perspectives in intellectual development: The Minnesota Symposia on Child Psychology* (Vol. 19, pp. 159-194). Hillsdale, NJ: Erlbaum.
- Resnick, L. B., & Ford, W. F. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Reys, R.E. (1984). Mental computation and estimation: Past, present and future. *Elementary School Journal*, 84, 547-557.
- Reys, R.E., Rybolt, J.F., Bestgen, B.J., & Wyatt, J.W. (1982, May). Processes used by good computational estimators. *Journal for Research in Mathematics Education*, 13, 183-201.

- Richards, F. (1978). *The numerical needs of the school leaver*. London, England: Hackney Teachers' Centre.
- Robitaille, D. F. (1981). *The 1981 B.C. Mathematics Assessment: Summary report*. Victoria, B.C.: Learning Assessment Branch, Ministry of Education.
- Robitaille, D.F. & O'Shea, T.J. (1985, October). *The 1985 British Columbia Mathematics Assessment: General report*. Victoria, B.C.: Learning Assessment Branch, Ministry of Education.
- Robitaille, D.F., O'Shea, T.J., & Dirks, M.K. (1982). *The Second International Mathematics Study: The teaching and learning of mathematics in British Columbia - Mathematics 8 and Algebra 12*. Victoria, B.C.: Learning Assessment Branch, Ministry of Education.
- Robitaille, D.F. & Sherrill, J.M. (1977). *The B.C. Mathematics Assessment: Test results*. Victoria, B.C.: Learning Assessment Branch, Ministry of Education.
- Ross, S.H. (1989, February). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher*, 36(6), 47-51.
- Russell, B. (1967). *History of western philosophy*. London: George Allen & Unwin Ltd.
- Russell, S.J., & Friel, S.N. (1989). Collecting and analyzing real data in the elementary school classroom. In R. Trafton (Ed.), *New directions for elementary school mathematics: 1989 yearbook* (pp. 134-148). Reston, VA: National Council of Teachers of Mathematics.
- Ryan, A., Sackney, L., & Birnie, H. (1981). *Program delivery in the small high school*. Regina: Saskatchewan School Trustees Association.
- Saskatchewan Education (1969). *The new Division III mathematics curriculum: Grades VII, VIII, and IX*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1977). *A curriculum guide for Division IV Mathematics 10, 20, 30*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1978, September). *A curriculum guide for Division III mathematics*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1981, September). *A curriculum guide for Division I and II mathematics*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1984). *Directions*. Regina, SASK: Saskatchewan Education.

- Saskatchewan Education (1984, August). *A curriculum guide for Division IV: Algebra 10, 20, 30 and Geometry-Trigonometry 10, 20, 30 - Interim Edition*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1986). *Program policy proposals: Report of the Core Advisory Committee*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1987, Decemeber). *Core curriculum: Plans for Implementation*. Regina, SASK: Saskatchewan Education.
- Saskatchewan Education (1988, August). *Understanding the Common Essential Learnings: A handbook for teachers*. Regina, SASK: Saskatchewan Education.
- Saskatoon Board of Education (1986, June). *Grade nine algebra: Course outline*. Saskatoon, SASK: Saskatoon Board of Education.
- Sawyer, W.W. (1989). Vision in elementary mathematics. *Mathematics in School*, 18(2), 6-7.
- Schoenfeld, A.H., & Arcavi, A. (1988, September). On the meaning of a variable. *Mathematics Teacher*, 81(6), 420-427.
- Scott, D.A., & Stephenson, J.W. (1985). *Preliminary analysis of failure rates in Math 175.6*. Unpublished paper. Saskatoon, SASK: University of Saskatchewan.
- Scribner, S. (1985). Knowledge at work. *Anthropology and Education Quarterly*, 16, 199-206.
- Shulte, A.P. (1982). Finding and using data: A basic skill. In S. Hill (Ed.), *Education in the 80's: Mathematics* (pp. 65-72). Washington, D.C. National Education Association.
- Sigurdson, S.E. (1970). Mathematics education in Western Canada. In P.S. Jones (Ed.), *A history of mathematics education in the United States and Canada* (pp. 412-425). Reston, VA: National Council of Teachers of Mathematics.
- Simon, M.A., & Stimson, V.C. (1988). Developing algebraic representation using diagrams. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 136-141). Reston, VA: National Council of Teachers of Mathematics.
- Smith, A.D. (1978). *Generic skills: Keys to job performance*. Ottawa, ONT: Canada Employment and Immigration Commission. (Eric microfiche ED220578)

- Smith, M.K. (1989, February). Why is Pythagoras following me? *Phi Delta Kappan*, 446-454.
- Sobel, M.A., & Maletsky, E.M. (1988). *Teaching mathematics: A sourcebook of aids, activities, and strategies*. Englewood Cliffs, NJ: Prentice-Hall.
- Sowder, J.T., & M.M. Wheeler (1989, March). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education*, 20(2), 130-146.
- Stanley, J.C., & Benbow, C.P. (1982, May). Educating mathematically precocious youth: Twelve policy recommendations. *Educational Researcher*, 11(5), 4-9.
- Stanley, J.C., & Benbow, C.P. (1983). SMPY's first decade: Ten years of posing problems and solving them. *Journal of Special Education*, 17(1), 11-25.
- Steffe, L.P., & Hirstein, J.J. (1976). Children's thinking in measurement situations. In D. Nelson & R.E. Reys (Eds.), *Measurement in school mathematics* (pp. 35-59). Reston, VA: National Council of Teachers of Mathematics.
- Stigler, J.W., Fuson, K.C., Ham, M., & Kim M.S. (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbooks. *Cognition and Instruction*, 3(3), 153-171.
- Sutherland, R. (1987). A study of the use of understanding of algebra related topics within a Logo environment. In J.C. Bergeron, N. Herscovics, & C. Kieren (Eds.), *Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education* (Vol. 1, pp. 241-247). Montreal, Quebec: Universite de Montreal.
- Swift, J. (1983). A statistics course to lighten the information overload. In G. Shufelt (Ed.), *The agenda in action: 1983 NCTM Yearbook* (pp. 110-119). Reston, VA: National Council of Teachers of Mathematics.
- Swinson, K. *Survey of employer needs*. Queensland, Australia: Board of Secondary School Studies.
- Thompson, C. S., & Hendrickson, A. D. (1986, March). Verbal addition and subtraction problems: Some difficulties and solutions. *Arithmetic Teacher*, 33(7), 21-25.
- Thompson, F.M. (1988). Algebraic instruction for the younger child. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 69-77). Reston, VA: National Council of Teacher of Mathematics.
- Thorndike, E. L. (1922). *The psychology of arithmetic*. New York: MacMillan.

- Thorpe, J.A. (1989). Algebra: What should we teach and how should we teach it? In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 11-24). Reston, VA: National Council of Teachers of Mathematics.
- Travers, K.J., Pikaart, L., Suydam, M.N., & Runion, G.E. (1977). *Mathematics teaching*. New York: Harper & Row.
- University of Saskatchewan (Spring, 1989). *Issues and Options: Accessibility, equity, and higher learning for the people of Saskatchewan*. Saskatoon, SASK: University of Saskatchewan.
- Usiskin, Z. (1980). What should not be in the algebra and geometry curricula of average college-bound students? *Mathematics Teacher*, 73, 413-424.
- Usiskin, Z. (1987). Resolving the continuing dilemmas in geometry. In M.M. Lindquist (Ed.), *Learning and teaching geometry, K - 12* (pp. 17-31). Reston, VA: National Council of Teachers of Mathematics.
- Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 8-19). Reston, VA: National Council of Teacher of Mathematics.
- Varga, T. (1988, August). Mathematics education in Hungary today. *Educational Studies in Mathematics*, 291-298.
- Weaver, F.S. (1982, Winter). Introductory statistics and general education. *The Journal of General Education*, 33(4), 287-294.
- Wheatley, G.H. (1980, December). Calculators in the classroom. *Arithmetic Teacher*, 28(4), 37-39.
- Wheeler, D. (1989). Contexts for research on the teaching and learning of algebra. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 278-287). Reston, VA: National Council of Teachers of Mathematics.
- Wyatt, J. W. (1979, November). The status of hand-held calculators in school. *Phi Delta Kappan*, 61(3), 217-218.